Minors and Cofactors.

(1) **Minor of an element:** If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}

Consider the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of minors $M =$

$\begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}$,

Where M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, M_{12} = minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

M_{13} = Minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly, we can find the minors of other elements. Using this concept the value of determinant can be

$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

or, $\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$ or, $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$.

(2) **Cofactor of an element:** The cofactor of an element a_{ij} (i.e. the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij} or A_{ij} or F_{ij}.

$C_{ij} = (-1)^{i+j} M_{ij}$

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of cofactors is $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$, where

$C_{11} = (-1)^{1+1} M_{11} = +M_{11}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$ and $C_{13} = (-1)^{1+3} M_{13} = +M_{13}$

Similarly, we can find the cofactors of other elements.

Note:The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant i.e. $\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

Where the capital letters C_{11}, C_{12}, C_{13} etc. denote the cofactors of a_{11}, a_{12}, a_{13} etc.

❑In general, it should be noted

$a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3} = 0, \text{ if } i \neq j$ or $a_{1i}C_{1j} + a_{2i}C_{2j} + a_{3i}C_{3j} = 0, \text{ if } i \neq j$

❑If Δ' is the determinant formed by replacing the elements of a determinant Δ by their corresponding cofactors, then if $\Delta = 0$, then $\Delta^C = 0$, $\Delta' = \Delta^{n-1}$, where n is the order of the determinant.