Application of Determinants in solving a system of Linear Equations.

Consider a system of simultaneous linear equations is given by
$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad .....(i)$$

A set of values of the variables x, y, z which simultaneously satisfy these three equations is called a solution. A system of linear equations may have a unique solution or many solutions, or no solution at all, if it has a solution (whether unique or not) the system is said to be consistent. If it has no solution, it is called an inconsistent system.

If $d_1 = d_2 = d_3 = 0$ in (i) then the system of equations is said to be a homogeneous system. Otherwise it is called a non-homogeneous system of equations.

Theorem 1: (Cramer's rule) the solution of the system of simultaneous linear equations

$a_1x + b_1y = c_1$ .....(i) and $a_2x + b_2y = c_2$ .....(ii)

is given by $x = \frac{D_1}{D}, y = \frac{D_2}{D}$, where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, provided that $D \neq 0$

Note: Here $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the determinant of the coefficient matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

The determinant D_1 is obtained by replacing first column in D by the column of the right hand side

of the given equations. The determinant D_2 is obtained by replacing the second column in D by the right most column in the given system of equations.

(1) **Solution of system of linear equations in three variables by Cramer's rule:**

Theorem 2:(Cramer's Rule) the solution of the system of linear equations

$a_1x + b_1y + c_1z = d_1$ .....(i)

$a_2x + b_2y + c_2z = d_2$ .....(ii)

$a_3x + b_3y + c_3z = d_3$ .....(iii)

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$, where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$, Provided that $D \neq 0$

Note: Here D is the determinant of the coefficient matrix. The determinant D_1 is obtained by replacing the elements in first column of D by d_1, d_2, d_3. D_2 is obtained by replacing the element in the second column of D by d_1, d_2, d_3 and to obtain D_3, replace elements in the third column of D by d_1, d_2, d_3.

Theorem 3:(Cramer's Rule) let there be a system of nsimultaneous linear equation n unknown given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{vmatrix}$ and let D_j, be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$. Then, $x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$,....., $x_n = \frac{D_n}{D}$, Provided that $D \neq 0$

(2) **Conditions for consistency**

Case 1: For a system of 2 simultaneous linear equations with 2 unknowns

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}, y = \frac{D_2}{D}$.

(ii) If $D = 0$ and $D_1 = D_2 = 0$, then the system is consistent and has infinitely many solutions.

(iii) If $D = 0$ and one of D_1 and D_2 is non-zero, then the system is inconsistent.

Case 2: For a system of 3 simultaneous linear equations in three unknowns

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent with infinitely many solutions.

(iii) If $D = 0$ and at least one of the determinants D_1, D_2, D_3 is non-zero, then given of equations is inconsistent.

(3) **Algorithm for solving a system of simultaneous linear equations by Cramer's rule (Determinant method)**

Step 1: Obtain $D,\ D_1, D_2$ and D_3

Step 2: Find the value of D. If $D \neq 0$, then the system of the equations is consistent has a unique solution. To find the solution, obtain the values of $D_1,\ D_2$ and D_3. The solutions is given by $x = \frac{D_1}{D},\ y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$. If $D = 0$ go to step 3.

Step 3: Find the values of D_1, D_2, D_3. If at least one of these determinants is non-zero, then the system is inconsistent. If $D_1 = D_2 = D_3 = 0,$ then go to step 4

Step 4: Take any two equations out of three given equations and shift one of the variables, say z on the right hand side to obtain two equations in x, y. Solve these two equations by Cramer's rule to obtain x, y, in terms of z.

Note: The system of following homogeneous equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$ is always consistent.

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then this system h

As the unique solution $x = y = z = 0$ known as **trivial**