Special Types of Matrices.

(1) **Symmetric and skew-symmetric matrix**

(i) **Symmetric matrix**: A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all *i, j* or $A^T = A$

Example: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Note: Every unit matrix and square zero matrix are symmetric matrices.

- Maximum number of different elements in a symmetric matrix is $\frac{n(n+1)}{2}$

(ii) **Skew-symmetric matrix**: A square matrix $A = [a_{ij}]$ is called skew- symmetric matrix if $a_{ij} = -a_{ji}$ for all *i, j*

or $A^T = -A$. *Example*: $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

Note: All principal diagonal elements of a skew- symmetric matrix are always zero because for any diagonal element. $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

- Trace of a skew symmetric matrix is always 0.

Properties of symmetric and skew-symmetric matrices:

(i) If *A* is a square matrix, then $A + A^T, AA^T, A^T A$ are symmetric matrices, while $A - A^T$ is skew-symmetric matrix.

(ii) If *A* is a symmetric matrix, then $-A, KA, A^T, A^n, A^{-1}, B^T AB$ are also symmetric matrices, where $n \in N$, $K \in R$ and *B* is a square matrix of order that of *A*

(iii) If A is a skew-symmetric matrix, then

(a) A^{2n} is a symmetric matrix for $n \in N$,

(b) A^{2n+1} is a skew-symmetric matrix for $n \in N$,

(c) *kA* is also skew-symmetric matrix, where $k \in R$,

(d) $B^T AB$ is also skew- symmetric matrix where *B* is a square matrix of order that of *A.*

(iv) If *A*, *B* are two symmetric matrices, then

(a) $A \pm B,\ AB + BA$ are also symmetric matrices,

(b) $AB - BA$ is a skew- symmetric matrix,

(c) *AB* is a symmetric matrix, when $AB = BA$.

(v) If *A,B* are two skew-symmetric matrices, then

(a) $A \pm B,\ AB - BA$ are skew-symmetric matrices,

(b) $AB + BA$ is a symmetric matrix.

(vi) If *A* a skew-symmetric matrix and *C* is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix *A* can uniquely be expressed as sum of a symmetric and skew-symmetric matrix *i.e.*

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right].$$

(2) **Singular and Non-singular matrix:** Any square matrix *A* is said to be non-singular if $|A| \neq 0$, and a square matrix *A* is said to be singular if $|A| = 0$. Here $|A|$ (or det(*A*) or simply det |*A*| means corresponding determinant of square matrix *A*.

Example: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2 \Rightarrow A$ is a non-singular matrix.

(3) **Hermitian and skew-Hermitian matrix:** A square matrix $A = [a_{ij}]$ is said to be hermitian matrix if $a_{ij} = \overline{a}_{ji}\ \ \forall i.j\ i.e.\ A = A^\theta$. *Example*: $\begin{bmatrix} a & b + ic \\ b - ic & d \end{bmatrix}, \begin{bmatrix} 3 & 3 - 4i & 5 + 2i \\ 3 + 4i & 5 & -2 + i \\ 5 - 2i & -2 - i & 2 \end{bmatrix}$ are Hermitian matrices.

Note: If *A* is a Hermitian matrix then $a_{ii} = \overline{a}_{ii} \Rightarrow a_{ii}$ is real $\forall i$, thus every diagonal element of a Hermitian matrix must be real.

❑ A Hermitian matrix over the set of real numbers is actually a real symmetric matrix and a square matrix, $A=|a_{ij}|$ is said to be a skew-Hermitian if $a_{ij} = -\bar{a}_{ji}. \forall i,j i.e. A^{\theta} = -A$.

Example: $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$ are skew-Hermitian matrices.

❑ If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$ i.e. a_{ii} must be purely imaginary or zero.

❑ A skew-Hermitian matrix over the set of real numbers is actually a real skew-symmetric matrix.

(4) **Orthogonal matrix:** A square matrix A is called orthogonal if $AA^T = I = A^T A$ *i.e.* if $A^{-1} = A^T$

Example: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal because $A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A^T$

In fact every unit matrix is orthogonal.

(5) **Idempotent matrix**: A square matrix A is called an idempotent matrix if $A^2 = A$.

Example: $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix, because $A^2 = \begin{bmatrix} 1/4+1/4 & 1/4+1/4 \\ 1/4+1/4 & 1/4+1/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A$.

Also, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are idempotent matrices because $A^2 = A$ and $B^2 = B$.

In fact every unit matrix is indempotent.

(6) **Involutory matrix:** A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix because $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

In fact every unit matrix is involutory.

(7) **Nilpotent matrix:** A square matrix A is called a nilpotent matrix if there exists a $p \in N$ such that $A^p = 0$

Example: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a nilpotent matrix because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ (Here $P = 2$)

(8) **Unitary matrix:** A square matrix is said to be unitary, if $\bar{A}' A = I$ since $|\bar{A}'| = |A|$ and $|\bar{A}' A| = |\bar{A}'||A|$ therefore if $\bar{A}' A$=I, we have $|\bar{A}'||A| = 1$

Thus the determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.

Hence $\bar{A}' A = I \Rightarrow A\bar{A}' = I$

(9) **Periodic matrix**: A matrix *A* will be called a periodic matrix if $A^{k+1} = A$ where *k* is a positive integer. If, however *k* is the least positive integer for which $A^{k+1} = A$, then *k* is said to be the period of *A.*

(10) **Differentiation of a matrix:** If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is a differentiation of matrix *A*.

Example: If $A = \begin{bmatrix} x^2 & \sin x \\ 2x & 2 \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} 2x & \cos x \\ 2 & 0 \end{bmatrix}$

(11) **Submatrix :** Let *A* be *m×n* matrix, then a matrix obtained by leaving some rows or columns or both, of *A* is called a sub matrix of *A. Example* :If $A' = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 2 \\ 2 & 5 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}$ are sub matrices of matrix $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 2 & 4 \\ 2 & 5 & 3 & 1 \end{bmatrix}$

(12) **Conjugate of a matrix:** The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\overline{A}$. *Example*: $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$

then $\overline{A} = \begin{bmatrix} 1-2i & 2+3i & 3-4i \\ 4+5i & 5-6i & 6+7i \\ 8 & 7-8i & 7 \end{bmatrix}$

Properties of conjugates:

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$

(iii) $\overline{(\alpha A)} = \overline{\alpha}\overline{A}, \alpha$ being any number

(iv) $\overline{(AB)} = \overline{A}\,\overline{B}, A$ and *B* being conformable for multiplication.

(13) **Transpose conjugate of a matrix:** The transpose of the conjugate of a matrix *A* is called transposed conjugate of A and is denoted by A^{θ}. The conjugate of the transpose of A is the same as the transpose of the conjugate of *A* *i.e.* $\overline{(A')} = (\overline{A})' = A^{\theta}$.

If $A = [a_{ij}]_{m\times n}$ then $A^{\theta} = [b_{ji}]_{n\times m}$ where $b_{ji} = \overline{a}_{ij}$ *i.e.* the $(j,i)^{th}$ element of A^{θ} = the conjugate of $(i,j)^{th}$ element of *A*.

Example: If $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$, then $A^{\theta} = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$

Properties of transpose conjugate

(i) $(A^{\theta})^{\theta} = A$

(ii) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$

(iii) $(kA)^{\theta} = \overline{K}A^{\theta}$, *K* being any number

(iv) $(AB)^{\theta} = B^{\theta}A^{\theta}$