Rank of Matrix

This lesson introduces the concept of **matrix rank** and explains how the rank of a matrix is revealed by its echelon form.

The Rank of a Matrix

You can think of an *r* x *c* matrix as a set of *r* row vectors, each having *c* elements; or you can think of it as a set of *c* column vectors, each having *r* elements.

The **rank** of a matrix is defined as (a) the maximum number of linearly independent *column* vectors in the matrix or (b) the maximum number of linearly independent *row* vectors in the matrix. Both definitions are equivalent.

For an *r* x *c* matrix,

- If *r* is less than *c*, then the maximum rank of the matrix is *r*.
- If *r* is greater than *c*, then the maximum rank of the matrix is *c*.

The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be one.

How to Find Matrix Rank

In this section, we describe a method for finding the rank of any matrix. This method assumes familiarity with echelon matrices and echelon transformations.

The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its row echelon matrix. Therefore, to find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of non-zero rows.

Consider matrix **A** and its row echelon matrix, $\mathbf{A}_{ref}$. Previously, we showed how to find the row echelon form for matrix **A**.

$$\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$\underset{\mathbf{A}}{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}} \qquad \underset{\mathbf{A}_{\mathbf{ref}}}{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

Because the row echelon form $\mathbf{A}_{\mathbf{ref}}$ has two non-zero rows, we know that matrix **A** has two independent row vectors; and we know that the rank of matrix **A** is 2.

You can verify that this is correct. Row 1 and Row 2 of matrix **A** are linearly independent. However, Row 3 is a linear combination of Rows 1 and 2. Specifically, Row 3 = 3*(Row 1) + 2*(Row 2). Therefore, matrix **A** has only two independent row vectors.

Full Rank Matrices

When all of the vectors in a matrix are linearly independent, the matrix is said to be **full rank**. Consider the matrices **A** and **B** below.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Notice that row 2 of matrix **A** is a scalar multiple of row 1; that is, row 2 is equal to twice row 1. Therefore, rows 1 and 2 are linearly dependent. Matrix **A** has only one linearly independent row, so its rank is 1. Hence, matrix **A** is not full rank.

Now, look at matrix **B**. All of its rows are linearly independent, so the rank of matrix **B** is 3. Matrix **B** is full rank.