System of Simultaneous Linear Equations:

Consider the following system of n linear equations in n unknowns:

$a_{11} x_1 + a_{12} x_2 + \dots\dots\dots + a_{1n} x_n = d_1$

$a_{21} x_1 + a_{22} x_2 + \dots\dots\dots + a_{2n} x_n = d_2$ . . .

$a_{n1} x_1 + a_{n2} x_2 + \dots\dots\dots + a_{nn} x_n = d_n$

This system of equation can be written in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} \cdots\cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots\cdots a_{2n} \\ \cdot & \vdots \qquad\qquad \vdots \\ a_{n1} & a_{n2} \cdots\cdots\cdots a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}_{n\times 1}$$

or AX = D.

The n × n matrix A is called the coefficient matrix of the system of linear equations.

Homogeneous and Non-Homogeneous System of Linear Equations

A system of equations AX = D is called a homogeneous system if D = O. Otherwise it is called a non-homogeneous systems of equations.

Solution of a System of Equations

Consider the system of equation AX = D.

A set of values of the variables x_1, x_2, ..., x_n which simultaneously satisfy all the equations is called a solution of the system of equations.

Consistent System

If the system of equations has one or more solutions, then it is said to be a consistent system of equations, otherwise it is an inconsistent system of equations.

Solution of a Non-Homogeneous System of Linear Equations

There are two methods of solving a non-homogeneous system of simultaneous linear equations.

(i) Cramer's Rule

(ii) Matrix Method

(i) Cramer's Rule:

It is discussed under the topic of Determinants.

(ii) Matrix Method:

Consider the equations

$a_1x + b_1y + c_1z = d_1$,

$a_2x + b_2y + c_2z = d_2$, ... (i)

$a_3x + b_3y + c_3z = d_3$.

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

then the equation (i) is equivalent to the matrix equation

A X = D. ...... (ii)

Multiplying both sides of (ii) by the inverse matrix A^{-1}, we get

$A^{-1}(AX) = A^{-1}D \Rightarrow IX = A^{-1}D$ $[\because A^{-1}A = I]$

$\Rightarrow X = A^{-1}D \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ ...... (iii)

where A_1, B_1 etc. are the cofactors of a_1, b_2 etc. in the determinant

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $(\Delta \neq 0)$.

(i) If A is a non-singular matrix, then the system of equations given by AX = D has a unique solutions given by $X = A^{-1}D$.

(ii) If A is a singular matrix, and (adjA)D = O, then the system of equations given by AX = D is consistent, with infinitely many solutions.

(iii) If A is a singular matrix, and (adjA)D ≠ O, then the system of equation given by AX = D is inconsistent.

Solution of Homogeneous System of Linear Equations:

Let AX = O be a homogeneous system of n linear equation with n unknowns. Now if A is non-singular then the system of equations will have a unique solution i.e. trivial solution and if A is singular then the system of equations will have infinitely many solutions.

Illustration:

If the system of equations $x + ay - z = 0$, $2x - y + az = 0$ and $ax + y + 2z = 0$ has a non trivial solution, then find the value of 'a'.

Solution:

Using $C_2 \to C_2 - aC_1$, $C_3 \to C_3 + C_1$, we get

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -(1+2a) & 2+a \\ a & 1-a^2 & 2+a \end{vmatrix} = 0.$$

=> $(2 + a)(-1 - 2a - 1 + a^2) = 0$

=> $(a + 2)(a^2 - 2a - 2) = 0$

=> $a = -2$, $a = 1 + \sqrt{3}$.

Illustration:

Find the value of 'k' for which the system of equations $(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has no solution.

Solution:

For the system of equations to have no solution, we must have

$(k+1)/k = 8/(k+3) \neq 4k/(3k-1)$

=> $(k + 1)(k + 3) = 8k$ and $8(3k - 1) \neq 4k(k + 3)$

=> $k^2 - 4k + 3 = 0$ => $k = 1, 3$.

For $= 1$, $8(3k - 1) = 16$ and $4k(k + 3) = 16$.

For $k = 3$, $8(3k - 1) = 64$ and $4k(k + 3) = 72$.

∴ for $k = 3$, $8(3k - 1) \neq 4k(k + 3)$.

∴ $k = 3$ is the required value of 'k' for no solution.