# Position of two points with respect to a given line

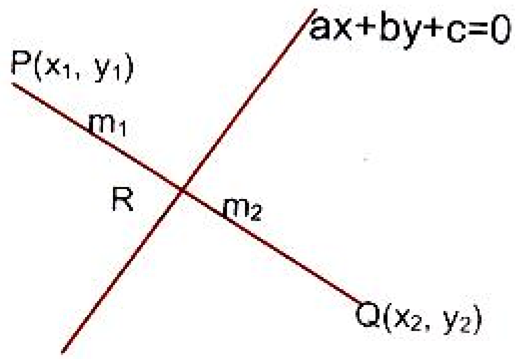
Let the line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points.
**Case 1:**
If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on the opposite sides of the line $ax + by + c = 0$, then the point R on the line $ax + by + c = 0$ divides the line PQ internally in the ratio $m_1 : m_2$, where $m_1/m_2$ must be positive.

Co-ordinates of R

are $(m_1x_2+m_2x_1/m_1+m_2,\ m_1y_2+m_2y_1/m_1+m_2)$.

Point R lies on the line $ax + by + c = 0$.

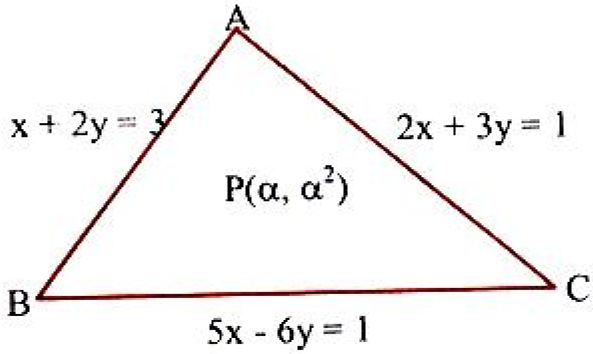
$\Rightarrow m_1/m_2 = ax_1+by_1+c/ax_2+by_2+c > 0$




So that $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ should have opposite signs.
**Case 2:**
If $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs then $m_1/m_2 = -ve$, so that the point R on the line $ax + by + c = 0$ will divide the line PQ externally in the ratio $m_1 : m_2$ and the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on the same side of the line $ax + by + c = 0$.

**Illustration:**
Find the range of θ in the interval (0, π) such that the points (3, 5) and (sinθ, cosθ) lie on the same side of the line $x + y - 1 = 0$.
**Solution:**
$3 + 5 - 1 = 7 > 0 \Rightarrow \sin\theta + \cos\theta - 1 > 0$
$\Rightarrow \sin(\pi/4 + \theta) > 1/\sqrt{2}$
$\Rightarrow \pi/4 < \pi/4 + \theta < 3\pi/4$
$\Rightarrow 0 < \theta < \pi/2$.

**Illustration:**
Find a, if $(\alpha, \alpha^2)$ lies inside the triangle having sides along the lines
$2x + 3y = 1$, $x + 2y - 3 = 0$, $6y = 5x - 1$.
**Solution:**
Let A, B, C be vertices of the triangle.
$A \equiv (-7, 5)$, $B \equiv (5/4, 7/8)$,
$C \equiv (1/3, 1/9)$.
Sign of A w.r.t. BC is –ve.

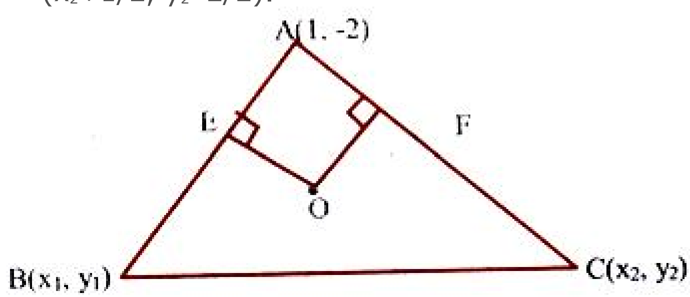



If p lies in-side the ¦ABC, then sign of P will be the same as sign of a w.r.t. the line BC
⇒ $5\alpha - 6\alpha^2 - 1 < 0$. …… (1)
Similarly $2\alpha + 3\alpha^2 - 1 > 0$. …… (2)
And, $\alpha + 2\alpha^2 - 3 < 0$. …… (3)
Solving, (1), (2) and (3) for α and then taking intersection,
We get α ? (1/2, 1) ∪ (−3/2, −1).

**Illustration:**

The equations of the perpendicular bisectors of the sides AB and AC of a triangle ABC are respectively $x - y + 5 = 0$ and $x + 2y = 0$. If the co-ordinates of A are (1, −2), find the equation of BC.

**Solution:**
From the figure,
$E \equiv (x_1+1/2, y_1-2/2)$,
$F \equiv (x_2+1/2, y_2-2/2)$.




**Alt text : Equations of the perpendicular bisectors of sides of triangle**

Since E and F lie on OE and OF respectively,

$x_1 - y_1 + 13 = 0$ … (1)
and $x_2 + 2y_2 - 3 = 0$ … (2)
Also, slope of AB = −1 and slope of AC is 2, so that
$x_1 + y_1 + 1 = 0$. … (3)
And $2x_2 - y_2 - 4 = 0$ … (4)
Solving these equations, we get the co-ordinates of B and C as
B ≡ (−7, 6) and C ≡ (11/5, 2/5)
⇒ Equation of BC is $14x + 23y - 40 = 0$.

**Illustration:**

Two fixed points A and B are taken on the co-ordinate axes such that OA = a and OB = b. Two variable points A′ and B′ are taken on the same axes such that OA′ + OB′ = OA + OB. Find the locus of the point of intersection of AB′ and A′B.

**Solution:**

Let A ≡ (a, 0), B (0, b), A′ ≡ (a′, 0), B′ ≡ (0, b′).

Equation of A′B is x/a' + y/b' = 1. …. (1)

and equation of AB′ is x/a + y/b' = 1. …. (2)

Subtracting (1) from (2), we get, x (1/a – 1/a') + y(1/b' – 1/b) = 0.

⇒ x(a'−a)/aa' + y(b−b')/bb' = 0. [Using a′ – a = b – b′]

⇒ x/a(b−b'+a) + y/bb', 0 ⇒ b′ = a(a+b)y/ay−bx. ….. (3)

From (2) b′x + ay = (4) we get x + y = a + b

which is the required locus.