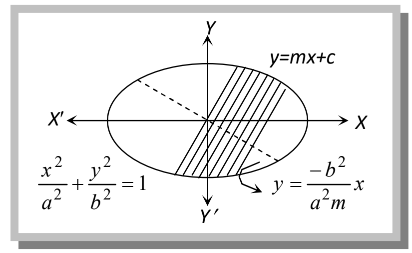
Diameter of the Ellipse.

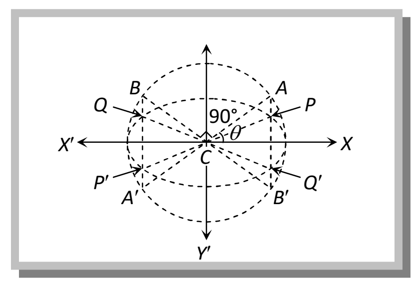
Definition: The locus of the mid- point of a system of parallel chords of an ellipse is called a diameter and the chords are called its double ordinates i.e. A line through the centre of an ellipse is called a diameter of the ellipse.
The point where the diameter intersects the ellipse is called the vertex of the diameter.

Equation of a diameter to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$: Let $y=mx+c$ be a system of parallel chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where m is a constant and c is a variable.



The equation of the diameter bisecting the chords of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=-\frac{b^2}{a^2m}x$, which is passing through (0, 0).

Conjugate diameter: Two diameters of an ellipse are said to be conjugate diameter if each bisects all chords parallel to the other.
Conjugate diameter of circle i.e. AA' and BB' are perpendicular to each other. Hence, conjugate diameter of ellipse are PP' and QQ'. Hence, angle between conjugate diameters of ellipse $>90^o$.
Now the coordinates of the four extremities of two conjugate diameters are
$P(a\cos\phi, b\sin\phi)$; $P'(-a\cos\phi, -b\sin\phi)$; $Q(-a\sin\phi, b\cos\phi)$; $Q'(a\sin\phi, -b\cos\phi)$

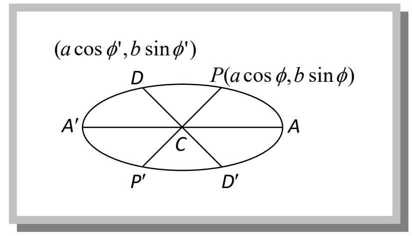


If $y=m_1x$ and $y=m_2x$ be two conjugate diameters of an ellipse, then $m_1m_2=\frac{-b^2}{a^2}$

(1) **Properties of diameters**
(i) The tangent at the extremity of any diameter is parallel to the chords it bisects or parallel to the conjugate diameter.
(ii) The tangent at the ends of any chord meet on the diameter which bisects the chord.

(2) **Properties of conjugate diameters**
(i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle,

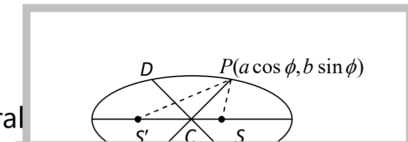
i.e. $\phi - \phi' = \frac{\pi}{2}$



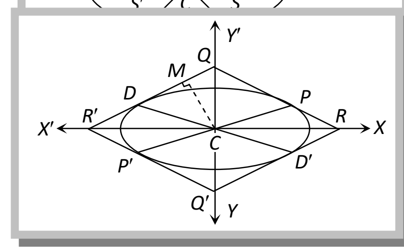
(ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi axes of the ellipse, i.e. $CP^2 + CD^2 = a^2 + b^2$

(iii) The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point,

i.e., $SP.S'P = CD^2$



(iv) The tangents at the extremities of a pair of conjugate diameters form a paral area is constant and equal to product of the axes, i.e.



Area of parallelogram $= (2a)(2b) =$ Area of rectangle contained under major and minor axes.

(v) The polar of any point with respect to ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid-pointis (x_1, y_1), i.e. chord is $T = S_1$.

(3) **Equi-conjugate diameters:** Two conjugate diameters are called equi-conjugate, if their lengths are equal i.e. $(CP)^2 = (CD)^2$

$\therefore a^2\cos^2\phi + b^2\sin^2\phi = a^2\sin^2\phi + b^2\cos^2\phi$

$\Rightarrow a^2(\cos^2\phi - \sin^2\phi) - b^2(\cos^2\phi - \sin^2\phi) = 0 \Rightarrow (a^2 - b^2)(\cos^2\phi - \sin^2\phi) = 0$

$\because (a^2 - b^2) \neq 0$, $\therefore \cos 2\phi = 0$. So, $\phi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

$\therefore \quad (CP) = (CD) = \sqrt{\frac{(a^2 + b^2)}{2}}$ forequi-conjugate diameters.

Important Tips

☞ If the point of intersection of the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ be at the extremities of the conjugate diameters of the former, then $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 2$

☞ The sum of the squares of the reciprocal of two perpendicular diameters of an ellipse is constant.

☞ In an ellipse, the major axis bisects all chords parallel to the minor axis and vice-versa, therefore major and minor axes of an ellipse are conjugate diameters of the ellipse but they do not satisfy the condition $m_1 . m_2 = -b^2 / a^2$ and are the only perpendicular conjugate diameters.