Rectangular or Equilateral Hyperbola.

(1) **Definition:** A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$.
The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0$, $h^2 > ab$ and coefficient of x^2 + coefficient of $y^2 = 0$

The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $y = \pm \frac{b}{a} x$.

The angle between these two asymptotes is given by $\tan\theta = \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \frac{b}{a}\left(\frac{-b}{a}\right)} = \frac{2b/a}{1 - b^2/a^2} = \frac{2ab}{a^2 - b^2}$.

If the asymptotes are at right angles, then $\theta = \pi/2 \Rightarrow \tan\theta = \tan\frac{\pi}{2} \Rightarrow \frac{2ab}{a^2 - b^2} = \tan\frac{\pi}{2}$

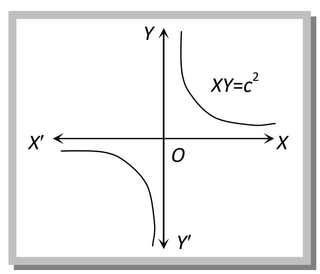
$\Rightarrow a^2 - b^2 = 0 \Rightarrow a = b \Rightarrow 2a = 2b$. Thus the transverse and conjugate axis of a rectangular hyperbola are equal and the equation is $x^2 - y^2 = a^2$. The equations of the asymptotes of the rectangular hyperbola are $y = \pm x$ *i.e.*, $y = x$ and $y = -x$. Clearly, each of these two asymptotes is inclined at 45° to the transverse axis.

(2) **Equation of the rectangular hyperbola referred to its asymptotes as the axes of coordinates:** Referred to the transverse and conjugate axis as the axes of coordinates, the equation of the rectangular hyperbola is

$x^2 - y^2 = a^2$ .....(i)

The asymptotes of (i) are $y = x$ and $y = -x$. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis, So, if we rotate the coordinate axes through an angle of $-\pi/4$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola and

$x = X\cos(-\pi/4) - Y\sin(-\pi/4) = \frac{X+Y}{\sqrt{2}}$ and $y = X\sin(-\pi/4) + Y\cos(-\pi/4) = \frac{Y-X}{\sqrt{2}}$.



Substituting the values of x and y in (i),

We obtain the $\left(\frac{X+Y}{\sqrt{2}}\right)^2 - \left(\frac{Y-X}{\sqrt{2}}\right)^2 = a^2 \Rightarrow XY = \frac{a^2}{2} \Rightarrow XY = c^2$

Where $c^2 = \frac{a^2}{2}$.

This is transformed equation of the rectangular hyperbola (i).

(3) **Parametric co-ordinates of a point on the hyperbola $XY = c^2$:** If t is non–zero variable, the coordinates of any point on the rectangular hyperbola $xy = c^2$ can be written as $(ct, c/t)$. The point $(ct, c/t)$ on the hyperbola $xy = c^2$ is generally referred as the point 't'.

For rectangular hyperbola the coordinates of foci are $(\pm a\sqrt{2}, 0)$ and directricesare $x = \pm a\sqrt{2}$.

For rectangular hyperbola $xy = c^2$, the coordinates of foci are $(\pm c\sqrt{2}, \pm c\sqrt{2})$ and directrices are $x + y = \pm c\sqrt{2}$.

(4) **Equation of the chord joining points t_1 and t_2 :** The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2$ is

$$y - \frac{c}{t_1} = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}(x - ct_1) \Rightarrow x + y\, t_1 t_2 = c(t_1 + t_2).$$

(5) **Equation of tangent in different forms**

(i) **Point form:** The equation of tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$

(ii) **Parametric form :** The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$ .On replacing x_1 by ct and y_1 by $\frac{c}{t}$ on the equation of the tangent at (x_1, y_1) i.e. $xy_1 + yx_1 = 2c^2$ we get $\frac{x}{t} + yt = 2c$.

Note: Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$

(6) **Equation of the normal in different forms:**

(i) **Point form :** The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$. As discussed in the equation of the tangent, we have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1}{x_1}$

So, the equation of the normal at (x_1, y_1) is $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1) \Rightarrow y - y_1 = \frac{x_1}{y_1}(x - x_1)$

$$\Rightarrow yy_1 - y_1^2 = xx_1 - x_1^2 \Rightarrow xx_1 - yy_1 = x_1^2 - y_1^2$$

This is the required equation of the normal at (x_1, y_1).

(ii) **Parametric form:** The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$. On replacing x_1 by ct and y_1 by c/t in the equation.

We obtain $xx_1 - yy_1 = x_1^2 - y_1^2$, $xct - \frac{yc}{t} = c^2t^2 - \frac{c^2}{t^2} \Rightarrow xt^3 - yt - ct^4 + c = 0$

Note: The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree in *t*. So, in general, four normals can be drawn from a point to the hyperbola $xy = c^2$

If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again in 't'' then; $t' = \frac{-1}{t^3}$.

Point of intersection of normals at 't_1' and 't_2' is $\left(\frac{c\{t_1t_2(t_1^2 + t_1t_2 + t_2^2) - 1\}}{t_1t_2(t_1 + t_2)}, \frac{c\{t_1^3t_2^3 + (t_1^2 + t_1t_2 + t_2^2)\}}{t_1t_2(t_1 + t_2)}\right)$

Important Tips

- *A triangle has its vertices on a rectangular hyperbola; then the orthocentre of the triangle also lies on the same hyperbola.*
- *All conics passing through the intersection of two rectangular hyperbolas are themselves rectangular hyperbolas.*
- *An infinite number of triangles can be inscribed in the rectangular hyperbola* $xy = c^2$ *whose all sides touch the parabola* $y^2 = 4ax$ *.*