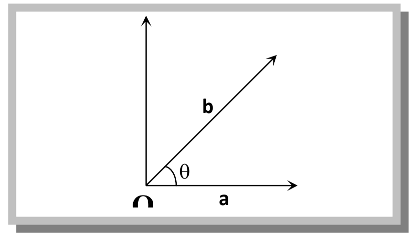
Vector or Cross product of Two Vectors.

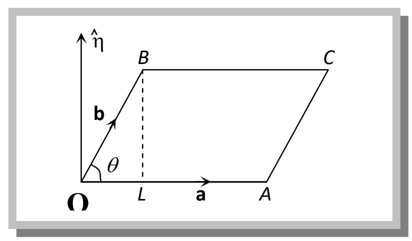
Let $\mathbf{a}, \mathbf{b}$ be two non-zero, non-parallel vectors. Then the vector product $\mathbf{a} \times \mathbf{b}$, in that order, is defined as a vector whose magnitude is $|\mathbf{a}||\mathbf{b}| \sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ whose direction is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ in such a way that $\mathbf{a}, \mathbf{b}$ and this direction constitute a right handed system.



In other words, $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\boldsymbol{\eta}}$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $\hat{\boldsymbol{\eta}}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \hat{\boldsymbol{\eta}}$ form a right handed system.

(1) **Geometrical interpretation of vector product:** If $\mathbf{a}, \mathbf{b}$ be two non-zero, non-parallel vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively and let θ be the angle between them. Complete the parallelogram *OACB*. Draw $BL \perp OA$.

In ΔOBL, $\sin\theta = \frac{BL}{OB} \Rightarrow BL = OB \sin\theta = |\mathbf{b}| \sin\theta$ ......(i)



Now, $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\boldsymbol{\eta}} = (OA)(BL)\hat{\boldsymbol{\eta}}$

$= (\text{Base} \times \text{Height})\, \hat{\boldsymbol{\eta}} = (\text{area of parallelogram } OACB)\, \hat{\boldsymbol{\eta}}$

= Vector area of the parallelogram *OACB*

Thus, $\mathbf{a} \times \mathbf{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\mathbf{a}$ and $\mathbf{b}$ as its adjacent sides and whose direction $\hat{\boldsymbol{\eta}}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \hat{\boldsymbol{\eta}}$ form a right handed system. Hence $\mathbf{a} \times \mathbf{b}$ represents the vector area of the parallelogram having adjacent sides along $\mathbf{a}$ and $\mathbf{b}$.

Thus, area of parallelogram *OACB*= $|\mathbf{a} \times \mathbf{b}|$.

Also, area of $\Delta OAB = \frac{1}{2}$ area of parallelogram *OACB* = $\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}|$

(2) **Properties of vector product**

(i) Vector product is not commutative *i.e.,* if $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$, however, $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(ii) If $\mathbf{a}, \mathbf{b}$ are two vectors and *m* is a scalar, then $m\mathbf{a} \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times m\mathbf{b}$

(iii) If $\mathbf{a}, \mathbf{b}$ are two vectors and *m, n* are scalars, then $m\mathbf{a} \times n\mathbf{b} = mn(\mathbf{a} \times \mathbf{b}) = m(\mathbf{a} \times n\mathbf{b}) = n(m\mathbf{a} \times \mathbf{b})$

(iv) Distributivity of vector product over vector addition.
Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three vectors. Then
(a) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (Left distributivity)

(b) $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$ (Right distributivity)

(v) For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we have $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$

(vi) The vector product of two non-zero vectors is zero vector *iff* they are parallel (Collinear) *i.e.,* $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} \parallel \mathbf{b}, \mathbf{a}, \mathbf{b}$ are non-zero vectors.

It follows from the above property that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every non-zero vector $\mathbf{a}$, which in turn implies that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

(vii) Vector product of orthonormal triad of unit vectors **i, j, k** using the definition of the vector product, we obtain $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

(viii) Lagrange's identity: If **a**, **b** are any two vector then $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} . \mathbf{b})^2$ or
$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} . \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$

(3) **Vector product in terms of components:** If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

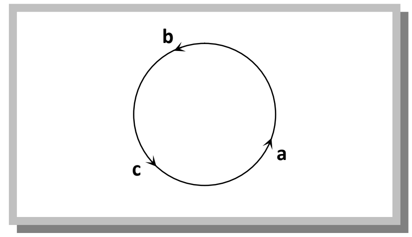
Then, $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

(4) **Angle between two vectors:** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$

Expression for $\sin\theta$ **:** If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and θ be angle between $\mathbf{a}$ and $\mathbf{b}$, then

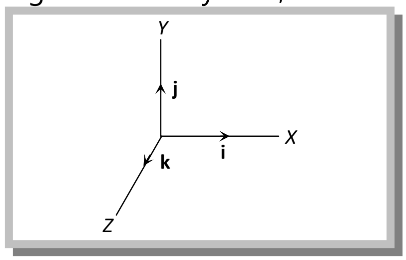
$$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

(5) (i) **Right handed system of vectors:** Three mutually perpendicular vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ form a right handed system of vector *iff* $\mathbf{a}\times\mathbf{b}=\mathbf{c},\ \mathbf{b}\times\mathbf{c}=\mathbf{a},\ \mathbf{c}\times\mathbf{a}=\mathbf{b}$



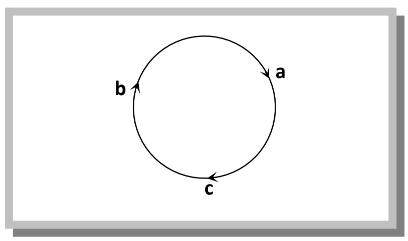
Example: The unit vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ form a right-handed system,

$\mathbf{i}\times\mathbf{j}=\mathbf{k},\mathbf{j}\times\mathbf{k}=\mathbf{i},\mathbf{k}\times\mathbf{i}=\mathbf{j}$



(ii) **Left handed system of vectors**: The vectors $\mathbf{a},\mathbf{b},\mathbf{c}$, mutually perpendicular to one another form a left handed system of vector *iff*

$\mathbf{c}\times\mathbf{b}=\mathbf{a},\mathbf{a}\times\mathbf{c}=\mathbf{b},\mathbf{b}\times\mathbf{a}=\mathbf{c}$



(6) **Vector normal to the plane of two given vectors:** If $\mathbf{a},\mathbf{b}$ be two non-zero, nonparallel vectors and let θ be the angle between them. $\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\boldsymbol{\eta}}$ Where $\hat{\boldsymbol{\eta}}$ is a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a},\mathbf{b},\boldsymbol{\eta}$ from a right-handed system.

$$\Rightarrow(\mathbf{a}\times\mathbf{b})=|\mathbf{a}\times\mathbf{b}|\hat{\boldsymbol{\eta}}\Rightarrow\hat{\boldsymbol{\eta}}=\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$$

Thus, $\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$. Note that $-\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is also a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$. Vectors of magnitude 'λ' normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ are given by $\pm\frac{\lambda(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$.