Vector Triple Product

The volume of a parallelepiped with sides **A**, **B** and **C** is the area of its base (say the parallelogram with area |**B** × **C**|) multiplied by its altitude, the component of **A** in the direction of **B** × **C.** This is the magnitude of **A • B** × **C;** but it is also the magnitude of the determinant of the matrix with columns **A**, **B** and **C**, so these linear functions of the vectors here are the same up to sign. This article is on Vector triple product. The usual sign convention gives

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \det(\mathbf{A}, \mathbf{B}, \mathbf{C})$$

This Vector Triple Product is not changed by cyclically permuting the vectors (for example to **B**, **C**, **A**) or by reversing the order of the factors in the dot product.

We can deduce then that **A • B** × **C** = **C • A** × **B** = **A** × **B • C.** In words, **we can switch the dot and cross product without changing anything in this entity.** (In either formula of course you must take the cross product first.) This product, like the determinant, changes sign if you just reverse the vectors in the cross product.

The vector triple product, **A** × (**B** × **C**) is a vector, is normal to **A** and normal to **B** × **C** which means it is in the plane of **B** and **C**. And it is linear in all three vectors.

We can deduce it is a multiple of **B** that is linear in **A** and **C** plus a multiple of **C** that is linear in **A** and **B**, with the condition that it is normal to **A**.

Any multiple of **B(A • C) – C(A • B)** will obey all these conditions.

What multiple is **A** × (**B** × **C**)?

Suppose **A** × (**B** × **C**) = **q(B(A • C) – C(A • B))** holds.

Earlier we saw that the square of the area of a parallelogram with sides **A** and **B** can be written either as **(A • A)(B • B) – (A • B)(A • B)** or (**B** × **A**) **•** (**B** × **A**). By interchanging the dot and first cross product on the right here you can rewrite this equality as

$$(\mathbf{B} \times \mathbf{A}) \cdot (\mathbf{B} \times \mathbf{A}) = \mathbf{B} \cdot (\mathbf{A} \times (\mathbf{B} \times \mathbf{A})) = (\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{A} \cdot \mathbf{B})$$

If we identify **A** with **C** in **A** × **(B** × **C)** and take the dot product of **A** × **(B** × **A)** with **B** we find q = 1, and we get

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

*This is sometimes called the **back cab** rule to make it easier to remember the appropriate signs. When using this name remember that the parentheses here are all as far back as possible in this expression The easiest way to get the signs right here without remembering anything is to guess a sign and then check it on the case **A** = **i** = **C**, **B** = **j**.*