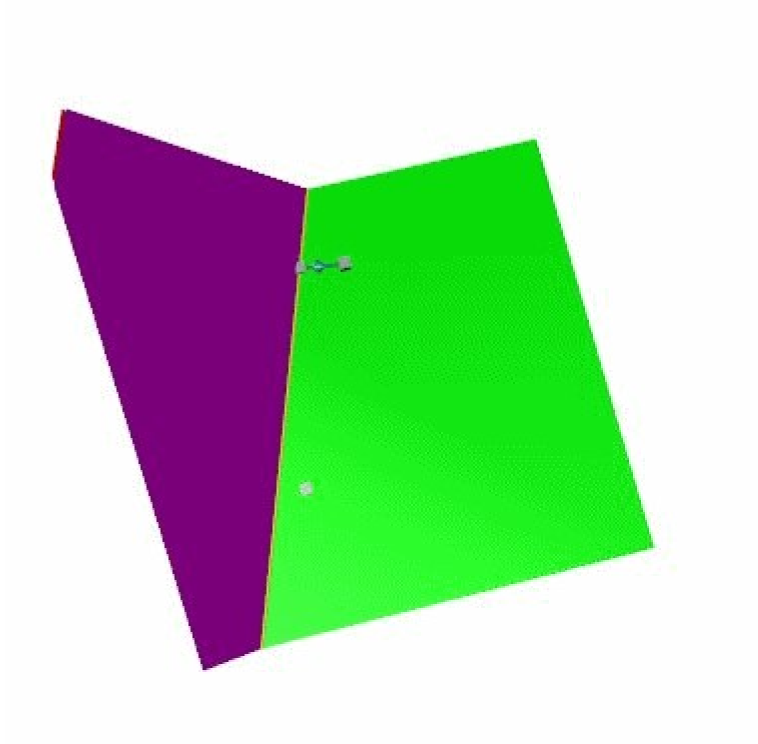
Vector Equations

The angle between two planes



The angle between two planes is found using the scalar product.
It is equal to the acute angle determined by the normal vectors of the planes.

Example

Calculate the angle between the planes

π_1: $x + 2y - 2z = 5$

and π_2: $6x - 3y + 2z = 8$

let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ represent the normal for π_1

and $\mathbf{b} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$ represent the normal for π_2

$$|\mathbf{a}| = \sqrt{1+4+4} = 3 \qquad |\mathbf{b}| = \sqrt{36+9+4} = 7$$

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}||\mathbf{b}|}$$

$$\cos\theta = \frac{1\times 6 - 2\times 3 - 2\times 2}{21}$$

$$\cos\theta = \frac{-4}{21}$$

$$\theta = 100.98^\circ \quad \text{i.e obtuse}$$

$$\theta = 79.02^\circ$$

The distance between parallel planes

Let P be a point on plane π_1 : $ax + by + cz = n$
$\mathbf{a.x} = n$

and Q be a point on plane π_2 : $ax + by + cz = m$
$\mathbf{a.x} = m$

Since the planes are parallel, they share the common normal, **a**
a=(a**i** +b**j**+c**k**)

The distance between the planes is

$$PQ = \frac{|m - n|}{|\mathbf{a}|}$$

Example

Calculate the distance between the planes

π_1: x + 2y - 2z = 5

and π_2: 6x + 12y - 12z = 8

$$x + 2y - 2z = 5$$

$$6x + 12y - 12z = 8$$

$$x + 2y - 2z = \frac{4}{3}$$

$$so \quad \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad n = 5 \quad \text{and} \quad m = \frac{4}{3}$$

$$PQ = \frac{|m-n|}{|\mathbf{a}|}$$

$$= \frac{\left|\frac{4}{3} - 5\right|}{\left|\sqrt{1+4+4}\right|}$$

$$= \frac{\frac{11}{3}}{3}$$

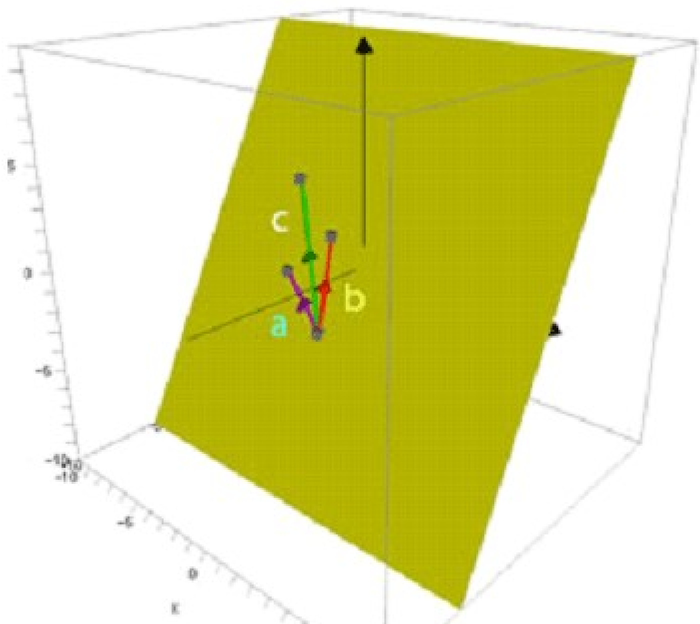
$$= \frac{11}{9}$$

$$= 1\frac{2}{9} \text{ units}$$

Coplanar vectors

If a relationship exists between the vectors **a**, **b** and **c** such that **c=**λ**a+**μ**b, where** λ and μ are constants**,**
then vectors a, b and c are co-planar.

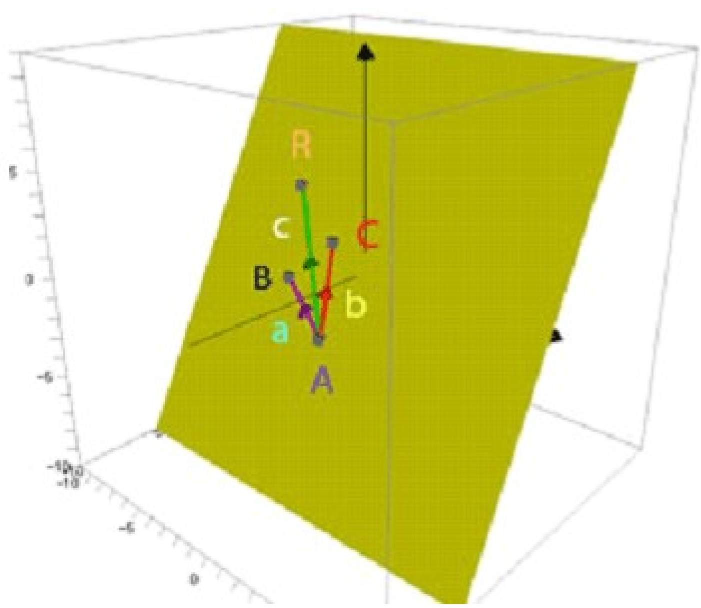
If three vectors are co-planar,
c=λ**a+**μ**b**



Vector equation of a plane

From the coplanar section above,
c=λa+μb

When position vectors are used,



$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$$

$$\overrightarrow{AR} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

$$\mathbf{r} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) + \mathbf{a}$$

$$\mathbf{r} = \lambda\mathbf{b} - \lambda\mathbf{a} + \mu\mathbf{c} - \mu\mathbf{a} + \mathbf{a}$$

$$\mathbf{r} = \mathbf{a} - \lambda\mathbf{a} - \mu\mathbf{a} + \mu\mathbf{c} + \lambda\mathbf{b}$$

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

r=(1-λ-u)**a**+ λ**b+**μ**c** is the **vector equation of the plane.**

Since λ and b are variable, there will be many possible equations for the plane.

Effects of changing λ and μ

Example

Find a vector equation of the plane through the points
A (-1,-2,-3) , B(-2,0,1) and C (-4,-1,-1)

$$\mathbf{r} = (1-\lambda-\mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

$$= (1-\lambda-\mu)\begin{pmatrix}-1\\-2\\-3\end{pmatrix} + \lambda\begin{pmatrix}-2\\0\\1\end{pmatrix} + \mu\begin{pmatrix}-4\\-1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-(1-\lambda-\mu)-2\lambda-4\mu\\-2(1-\lambda-\mu)-\mu\\-3(1-\lambda-\mu)+\lambda-\mu\end{pmatrix}$$

$$= \begin{pmatrix}-1+\lambda+\mu-2\lambda-4\mu\\-2+2\lambda+2\mu-\mu\\-3+3\lambda+3\mu+\lambda-\mu\end{pmatrix}$$

$$= \begin{pmatrix}-1-\lambda-3\mu\\-2+2\lambda+\mu\\-3+4\lambda+2\mu\end{pmatrix}$$

$$= (-1-\lambda-3\mu)\mathbf{i} + (-2+2\lambda+\mu)\mathbf{j} + (-3+4\lambda+2\mu)\mathbf{k}$$

If λ = 2 and μ =3

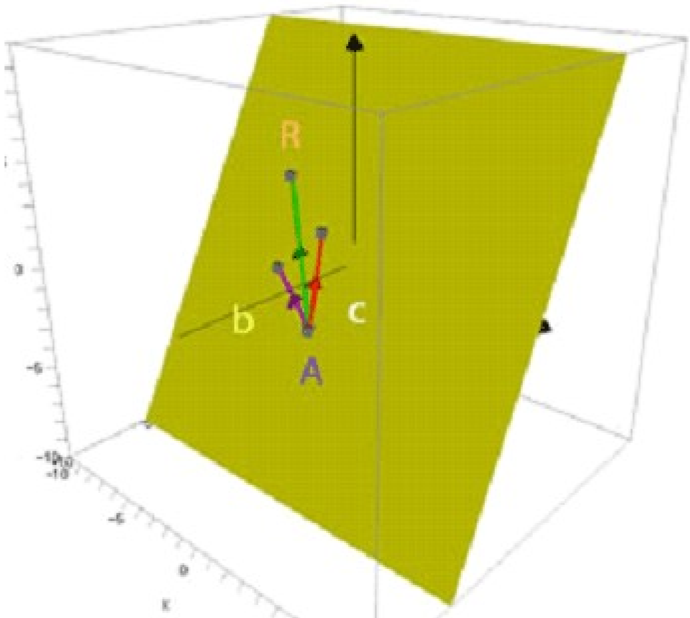
$$\mathbf{r} = (-1-\lambda-3\mu)\mathbf{i} + (-2+2\lambda+\mu)\mathbf{j} + (-3+4\lambda+2\mu)\mathbf{k}$$

$$\mathbf{r} = (-1-2-9)\mathbf{i} + (-2+4+3)\mathbf{j} + (-3+8+6)\mathbf{k}$$

$$\mathbf{r} = -12\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}$$

When A is a known point on the plane,
R is any old point on the plane and **b** and **c** are vectors parallel to the plane,

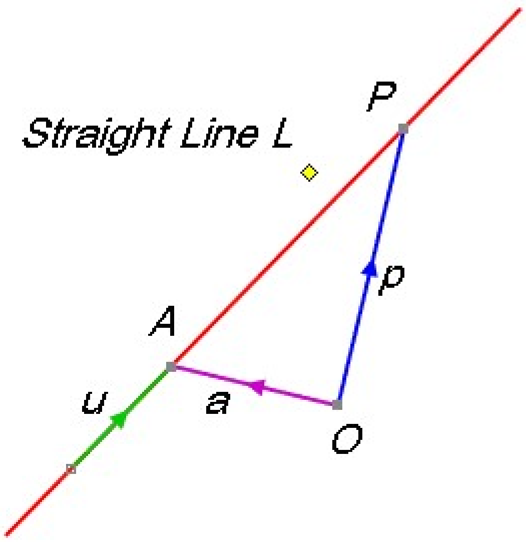
the **vector equation of the plane** is
r=a+λ**b+**μ**c**



The equations of a line

A line can be described when a point on it and its direction vector – a vector parallel to the line – are known.

In the diagram below, the line L passes through points $A(x_1,y_1,z_1)$ and P (x,y,z).



uis the direction vector a**i** +b**j** +c**k**
Being on the line, it has the same direction as any parallel line.

O is the origin.
a and **p** represent the position vectors of A and P.

P is on line L

$\Rightarrow \quad \overrightarrow{AP} = \lambda \mathbf{u}$ for some scalar λ

$\Rightarrow \quad \mathbf{p} - \mathbf{a} = \lambda \mathbf{u}$

$\Rightarrow \quad \mathbf{p} = \mathbf{a} + \lambda \mathbf{u}$

$$\mathbf{p} = \mathbf{a} + \lambda \mathbf{u}$$

is the vector equation of the line

convention often replaces **p** with **r**

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$

If two points are known, say A and B

then $\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\Rightarrow \quad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

$\Rightarrow \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} - \lambda\mathbf{a}$

$\Rightarrow \quad \mathbf{r} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$

In component form, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 + \lambda a \\ y_1 + \lambda b \\ z_1 + \lambda c \end{pmatrix}$$

giving the parametric equations

$$x = x_1 + \lambda a \quad , \quad y = y_1 + \lambda b \quad , \; z = z_1 + \lambda c$$

so

$$\frac{x - x_1}{a} = \lambda \qquad \frac{y - y_1}{b} = \lambda \qquad \frac{z - z_1}{c} = \lambda$$

Giving the symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

This is also known as :

standard form,

canonical form,

co-ordinate equation

Example

Find the vector equation of the straight line through (3,2,1) which is parallel to the vector 2**i** +3**j** +4**k**

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$$

$$\Rightarrow \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

are the vector equations of the line

Example

Find the vector form of the equation of the straight line which has parametric equations

$$x = 4 - 2\lambda \qquad y = 7 + \lambda \qquad z = 3 - 4\lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

$$\Rightarrow \ \mathbf{r} = 4\mathbf{i} + 7\mathbf{j} + 3\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$$

Example

Find the Cartesian form of the line which has position vector 3**i** +2**j** +**k** and is parallel to the vector **i** - **j** + **k**

$$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\Rightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore x = 3 + \lambda \qquad y = 2 - \lambda \qquad z = 1 + \lambda$$

$$\frac{x-3}{1} = \frac{y-2}{-1} = \frac{z-1}{1} = \lambda$$

$$\Rightarrow \quad x - 3 = 2 - y = z - 1 = \lambda$$

Example

Find the vector equation of the line passing through A(1,2,3) and B(4,5,6)

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$$

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$\Rightarrow \quad \mathbf{u} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\Rightarrow \quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

alternatively

$$\mathbf{r} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$$

$$\Rightarrow \mathbf{r} = (1-\lambda)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\Rightarrow \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\Rightarrow \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

Example

The vector equation of a line is

$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

State the point with z co-ordinate 3 which also lies on this line.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + 2\lambda \qquad y = 2 - \lambda \qquad z = 6 + 3\lambda$$

When $z = 3$

$$3 = 6 + 3\lambda$$

$$\Rightarrow \quad \lambda = \frac{3-6}{3} = -1$$

$$\Rightarrow \quad x = 3 - 2 = 1 \qquad y = 2 + 1 = 3 \qquad z = 6 - 3 = 3$$

$\Rightarrow$ point $(1,3,3)$ lies on line

Example

A line L has equations

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{3-z}{4}$$

Is the vector $\mathbf{s} = 6\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$ parallel to L ?

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{3-z}{4} = \lambda$$

$$\Rightarrow \quad x = -2 + 3\lambda \qquad y = 1 + 2\lambda \qquad z = 3 - 4\lambda$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$$

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$$

(-2,1,3) is a point on L

and $\lambda(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ is a direction vector.

s has direction ratio $6 : 4 : -8 = 3 : 2 : -4$

The direction ratios of **s** and **u** are the same

$\Rightarrow \quad \mathbf{s} \parallel \mathbf{u}$

The angle between a line and a plane

The angle θ between a line and a plane is the complement of the angle between the line and the normal to the plane.

If the line has direction vector **u** and the normal to the plane is **a,** then

$$\sin\theta^\circ = \left|\cos(90 - \theta)^\circ\right| = \frac{|\mathbf{a} \cdot \mathbf{u}|}{|\mathbf{a}||\mathbf{u}|}$$

Example

Given the equations

$$\frac{x-4}{3}=\frac{y-3}{2}=\frac{z-5}{6}$$

and the plane $6x+3y-2z=14$

1) Find the point of intersection

2) Find the angle the line makes with the plane.

1)

$$\frac{x-4}{3}=\frac{y-3}{2}=\frac{z-5}{6}=\lambda$$

$\Rightarrow \quad x=4+3\lambda \qquad y=3+2\lambda \qquad z=5+6\lambda$

$\therefore (\ 4+3\lambda\ ,\ 3+2\lambda\ ,5+6\lambda\ )$ lies on the plane

$$6(4+3\lambda)+3(3+2\lambda)-2(5+6\lambda)=14$$

$$24+18\lambda+9+6\lambda-10-12\lambda=14$$

$$23+12\lambda=14$$

$$\lambda=\frac{14-23}{12}=\frac{-3}{4}$$

$x=4+3\times\frac{-3}{4} \qquad y=3+2\times\frac{-3}{4} \qquad z=5+6\times\frac{-3}{4}$

$x=\frac{16-9}{4} \qquad y=\frac{12-6}{4} \qquad z=\frac{20-18}{4}$

$x=\frac{7}{4} \qquad y=\frac{3}{2} \qquad z=\frac{1}{2}$

The point of intersection is $\left(\frac{7}{4}, \frac{3}{2}, \frac{1}{2}\right)$

2)

$$\sin\theta^\circ = \left|\cos(90-\theta)^\circ\right| = \frac{|\mathbf{a}\bullet\mathbf{u}|}{|\mathbf{a}||\mathbf{u}|}$$

$$\mathbf{a} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$
$$\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

$$\sin\theta^\circ = \frac{\left|(6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})\bullet(3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})\right|}{\left(\left|\sqrt{36+9+4}\right|\right)\left|\left(\sqrt{9+4+36}\right)\right|}$$

$$\Rightarrow \quad \sin\theta^\circ = \frac{12}{49} \qquad (o \le \theta \le 90)$$
$$\Rightarrow \quad \theta = 14.175^\circ$$

The angle of intersection is 14.2°

The intersection of two lines

Example

Show that the lines with equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\frac{x+1}{12} = \frac{y-7}{6} = \frac{z-5}{3} = \lambda_2$$

intersect and find the point of intersection
and the equation of the plane
containing the lines.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + 4\lambda_1 \qquad y = 4 + \lambda_1 \qquad z = 1$$

and

$$\frac{x+1}{12} = \frac{y-7}{6} = \frac{z-5}{3} = \lambda_2$$

$$\Rightarrow \quad x = -1 + 12\lambda_2 \qquad y = 7 + 6\lambda_2 \qquad z = 5 + 3\lambda_2$$

Equating co-ordinates

$3+4\lambda_1 = -1+12\lambda_2 \qquad 4+\lambda_1 = 7+6\lambda_2 \qquad 1 = 5+3\lambda_2$

$4\lambda_1 = -4+12\lambda_2 \qquad (1)$

$\lambda_1 = 3+6\lambda_2 \qquad (2)$

$0 = 4+3\lambda_2 \qquad (3)$

From (3), $\quad 3\lambda_2 = -4$

$\Rightarrow \lambda_2 = \frac{-4}{3}$

$\Rightarrow \lambda_1 = 3+6\times\frac{-4}{3} = -5$

substituting

$$x = 3+4\lambda_1 \qquad y = 4+\lambda_1 \qquad z = 1$$

$$= 3-20 \qquad = 4-5$$

$$= -17 \qquad = -1$$

Intersection point is (-17, -1,1)

Let A(-17,-1,1) B(3,4,1) C(-1,7,5) be the points from the lines above

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ 4 \end{pmatrix}$$

$$\mathbf{n} \bullet \overrightarrow{\mathrm{AP}} = 0$$

Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{p} = \overrightarrow{OP}$, so $\overrightarrow{\mathrm{AP}} = \overrightarrow{OP} - \overrightarrow{OA}$

$\Rightarrow \quad \mathbf{n} \bullet (\mathbf{p} - \mathbf{a}) = 0$

Here, $\mathbf{n} = \left| \overrightarrow{AB} \times \overrightarrow{AC} \right|$

$$\mathbf{n} = \left| \overrightarrow{AB} \times \overrightarrow{AC} \right| = \left| \begin{pmatrix} i & j & k \\ 20 & 5 & 0 \\ 16 & 8 & 4 \end{pmatrix} \right|$$

$$= 20\mathbf{i} - 80\mathbf{j} + 80\mathbf{k}$$

$$= \mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{p} - \mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x+17 \\ y+1 \\ z-1 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{n} \bullet \overrightarrow{AP} = 0$$

$$\Rightarrow \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} x+17 \\ y+1 \\ z-1 \end{pmatrix} = 0$$

$$\Rightarrow x + 17 - 4(y+1) + 4(z-1) = 0$$

$$\Rightarrow x + 17 - 4y - 4 + 4z - 4 = 0$$

$$\Rightarrow x - 4y + 4z + 9 = 0$$

The intersection of two planes

To find the equations of the line of intersection
of two planes, a direction vector and point
on the line is required.

Since the line of intersection lies in both planes,
the direction vector is parallel to the vector products
of the normal of each plane.

Example

Find the equation for the line of intersection
of the planes

-3x + 2y + z = -5
7x + 3y - 2z = -2

$$-3x+2y+z=-5 \qquad\qquad 7x+3y-2z=-2$$

Let $z=0$

Then $\quad -3x+2y=-5.....(1)$

and $\quad 7x+3y=-2.....(2)$

$(2)\times 2 \quad 14x+6y=-4$

$(1)\times -3 \quad 9x-6y=15$

add $\quad 23x=11$

$$\Rightarrow \quad x=\frac{11}{23}$$

subst in (1)

$$-\frac{33}{23}+2y=-5$$

$$\Rightarrow \quad y=\frac{-5+\frac{33}{23}}{2}=\frac{-41}{23}$$

The point $\left(\frac{11}{23}, \frac{-41}{23}, 0\right)$ is on the line of intersection

Normal vectors are $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

and $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \left| \begin{pmatrix} i & j & k \\ -3 & 2 & 1 \\ 7 & 3 & -2 \end{pmatrix} \right|$$

$$= -7\mathbf{i} + \mathbf{j} - 23\mathbf{k}$$

$$\mathbf{r} = \begin{pmatrix} \frac{11}{23} \\ \frac{-41}{23} \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -7 \\ 1 \\ -23 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{23} \\ \frac{-41}{23} \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} \frac{7}{23} \\ \frac{-1}{23} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad x = \frac{11}{23} + \frac{7}{23}\lambda_1 \qquad y = \frac{-41}{23} - \frac{1}{23}\lambda_1 \qquad z = \lambda_1$$

The distance from a point to a plane

To find the distance of a point P to a plane

1. Find the equation of the projection PP′ by using the normal to the plane and the point P.
2. Find the co-ordinates of P′ , the intersection with the plane.
3. Apply the distance formula to PP′

Alternatively

The distance D between a point $P_0(x_0,y_0,z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Example

Find the distance between the point (3,1,-2) and the plane x +2y + 2z = - 4

$$\mathbf{r} = u + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + \lambda_1 \qquad y = 1 + 2\lambda_1 \qquad z = -2 + 2\lambda_1$$

Plane equation is $x+2y+2z+4=0$

$$\Rightarrow 3+\lambda_1 +2(1+2\lambda_1)+2(-2+2\lambda_1)+4=0$$

$$\Rightarrow 3+\lambda_1 +2+4\lambda_1 - 4+4\lambda_1+4=0$$

$$\Rightarrow 5+9\lambda_1 =0$$

$$\Rightarrow \lambda_1 = \frac{-5}{9}$$

$$\Rightarrow \quad x=3-\frac{5}{9} \qquad y=1-\frac{10}{9} \qquad z=-2-\frac{10}{9}$$

$$P'\left(\frac{22}{9}, -\frac{1}{9}, -\frac{28}{9}\right)$$

$$PP' = \begin{pmatrix} \frac{-5}{9} \\ -\frac{10}{9} \\ -\frac{10}{9} \end{pmatrix} = \frac{-5}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow |PP'| = \left|\frac{-5}{9}\sqrt{1+4+4}\right|$$

$$= \left|\frac{-5}{3}\right|$$

$$= \frac{5}{3} \text{ units}$$

Alternatively

$$x + 2y + 2z = -4 \qquad \text{at } (3,1,-2)$$

$$\Rightarrow \; x + 2y + 2z + 4 = 0$$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|3 + 2 - 4 + 4|}{\sqrt{1 + 4 + 4}}$$

$$= \frac{5}{3}$$

The distance is $\frac{5}{3}$ units

The distance from a point to a line

To find the distance of a point P to a Line L

1. Let the line have direction vector **u** and parameter λ
2. Find the co-ordinates of PP′ by using the scalar product with **u** and the point P.
3. Apply the distance formula to PP′

Find the distance between the line

$$\frac{x+3}{-6} = \frac{y-2}{9} = \frac{z+8}{6}$$

and the point P $(-1, 7, 4)$

$$P' = \begin{pmatrix} -3 \\ 2 \\ -8 \end{pmatrix} + \lambda_1 \begin{pmatrix} -6 \\ 9 \\ 6 \end{pmatrix}$$

$$\Rightarrow \quad x = -3 - 6\lambda_1 \qquad y = 2 + 9\lambda_1 \qquad z = -8 + 6\lambda_1$$

$P'(-3 - 6\lambda_1\ ,\ 2+9\lambda_1\ ,\text{-}8+6\lambda_1)$

$$\overrightarrow{PP'} = \begin{pmatrix} -3 - 6\lambda \\ 2 + 9\lambda_1 \\ -8 + 6\lambda_1 \end{pmatrix} - \begin{pmatrix} -1 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 - 6\lambda \\ -5 + 9\lambda \\ -12 + 6\lambda \end{pmatrix}$$

$$\overrightarrow{PP'} \bullet \mathbf{u} = 0$$

$$\Rightarrow \quad \begin{pmatrix} -2 - 6\lambda \\ -5 + 9\lambda \\ -12 + 6\lambda \end{pmatrix} \bullet \begin{pmatrix} -6 \\ 9 \\ 6 \end{pmatrix} = 0$$

$$\Rightarrow -6(-2 - 6\lambda) + 9(-5 + 9\lambda) + 6(-12 + 6\lambda) = 0$$

$$\Rightarrow 12 + 36\lambda - 45 + 81\lambda - 72 + 36\lambda = 0$$

$$\Rightarrow -105 + 153\lambda = 0$$

$$\Rightarrow \lambda = \frac{105}{153} = \frac{35}{51}$$

$$\overrightarrow{PP'} = \begin{pmatrix} -2-6\lambda \\ -5+9\lambda \\ -12+6\lambda \end{pmatrix} == \begin{pmatrix} -2-6\times\frac{35}{51} \\ -5+9\times\frac{35}{51} \\ -12+6\times\frac{35}{51} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-104}{17} \\ \frac{20}{17} \\ \frac{-134}{17} \end{pmatrix} == \frac{1}{17}\begin{pmatrix} -104 \\ 20 \\ -134 \end{pmatrix}$$

$$\Rightarrow PP' = \frac{1}{17}\sqrt{29172} = 10.04$$

The distance is 10.04 units

The intersection of three planes

To solve the intersection,
use the equations of the plane ax +by +cz +d = 0
to form an augmented matrix, which is solved
for x, y and z.

The intersection between three planes could be:

A single point

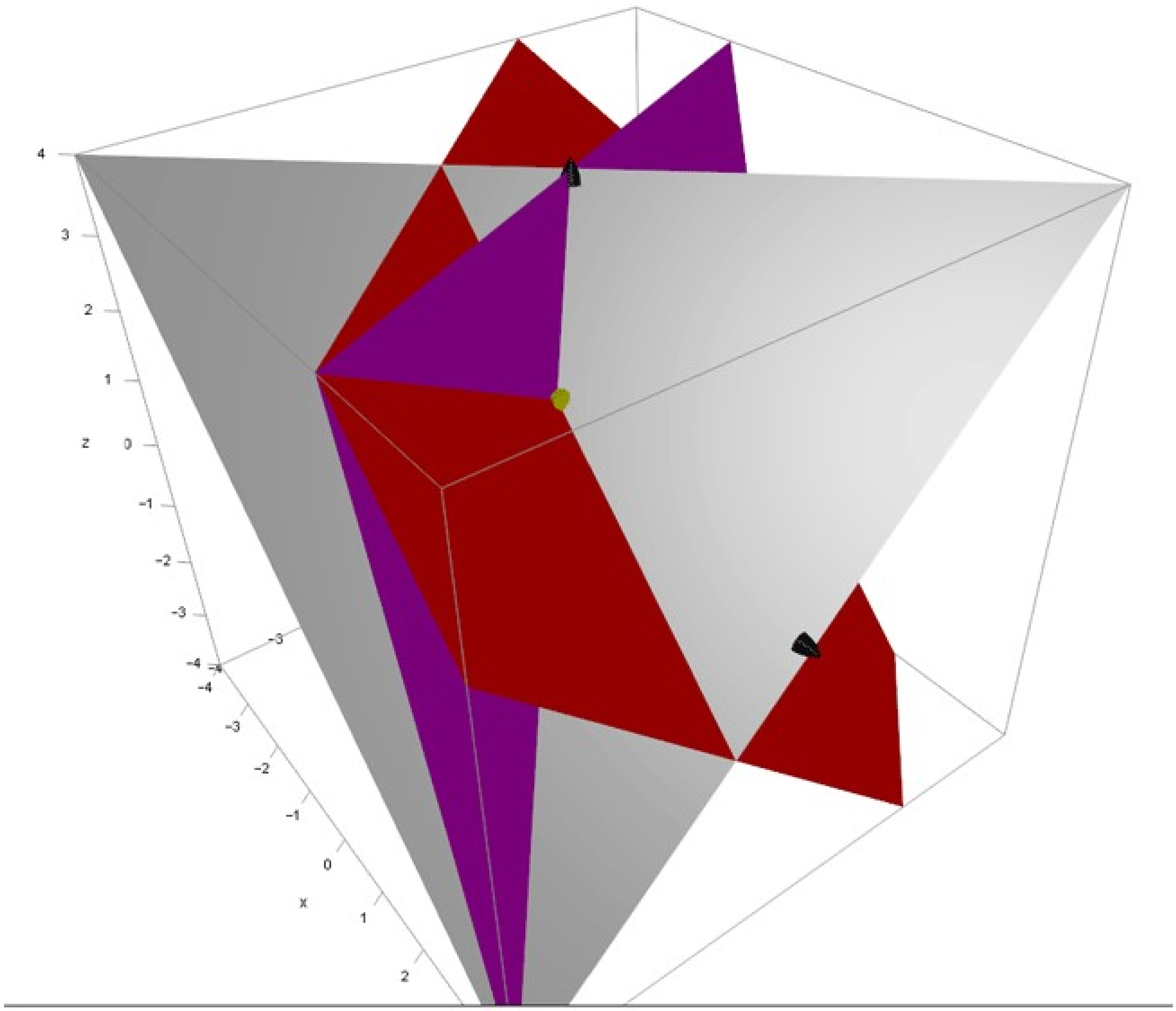
A unique solution is found

Example

$$x + y + z = 2$$
$$4x + 2y + z = 4$$
$$x - y + z = 4$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$Point \quad (1, -1, \; 2)$$



A line of intersection

An infinite number of solutions exist

Example

$$x + 2y + 2z = 11$$
$$x - y + 3z = 8$$
$$4x - y + 11z = 35$$

$$\begin{pmatrix} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

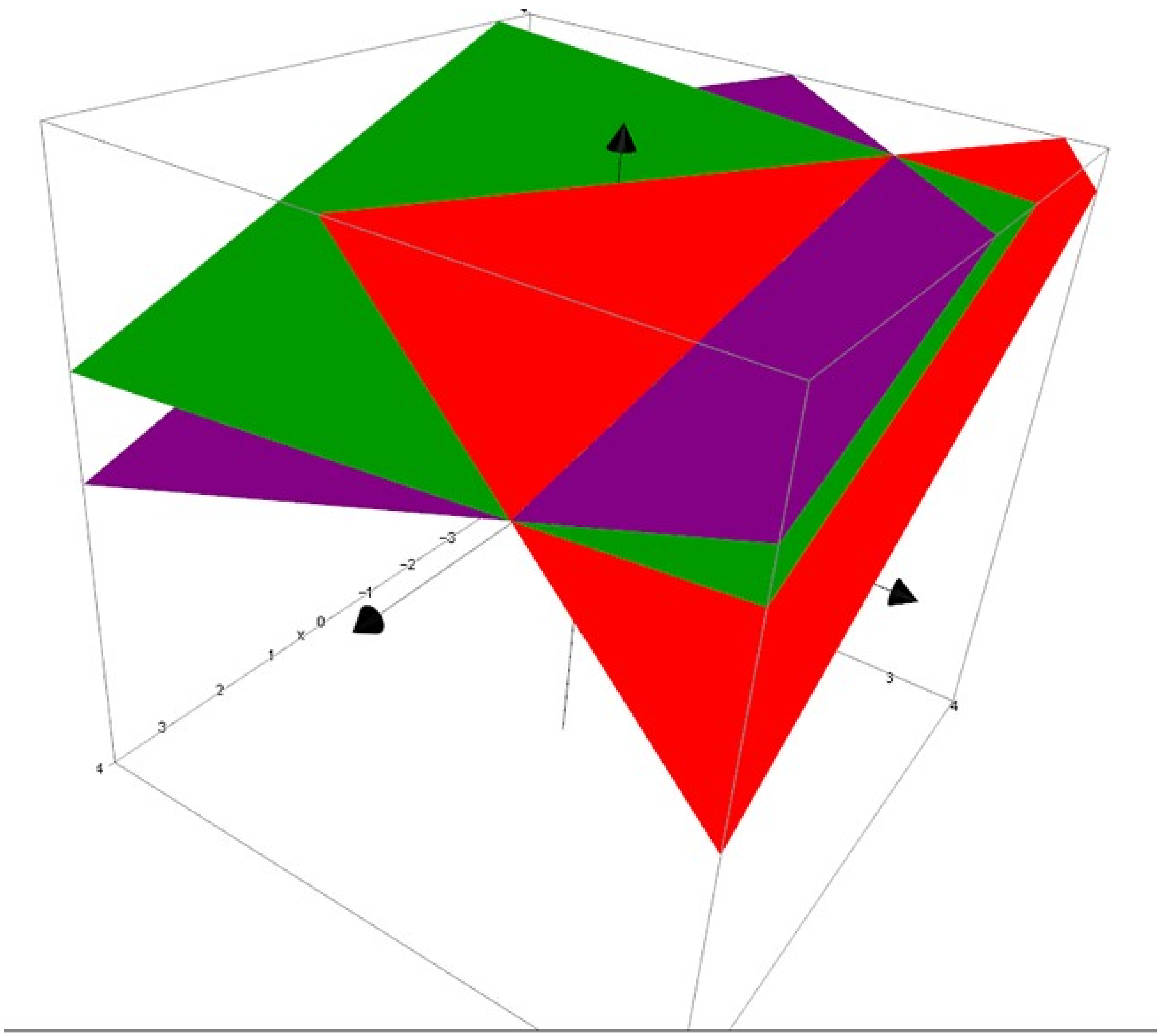
$$x + 2y + 2z = 11$$

$$x = 11 - 2y - 2z$$

$$x = 11 - 2\left(\frac{z+3}{3}\right) - 2z \qquad -3y + z = -3$$

$$x = 13 - \frac{8z}{3} = \frac{39 - 8z}{3} \quad \Rightarrow y = \frac{z+3}{3} \qquad z = z$$

Parametric equations



Two lines of intersection

An infinite number of solutions

Example

$$2x + 4y + 6z = 22$$
$$3y + 3z = -9$$
$$x + 2y + 3z = 16$$

which reduces to

$$\begin{pmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

The system is inconsistent

Using the second row

$$\text{let } z = t$$

$$so$$

$$y + t = -3$$

$$y = -3 - t$$

Substitute into first row

$$x + 2y + 3z = 11$$

$$x + 2(-3 - t) + 3t = 11$$

$$x - 6 - 2t + 3t = 11$$

$$x + t = 17$$

$$x = 17 - t$$

$$so$$

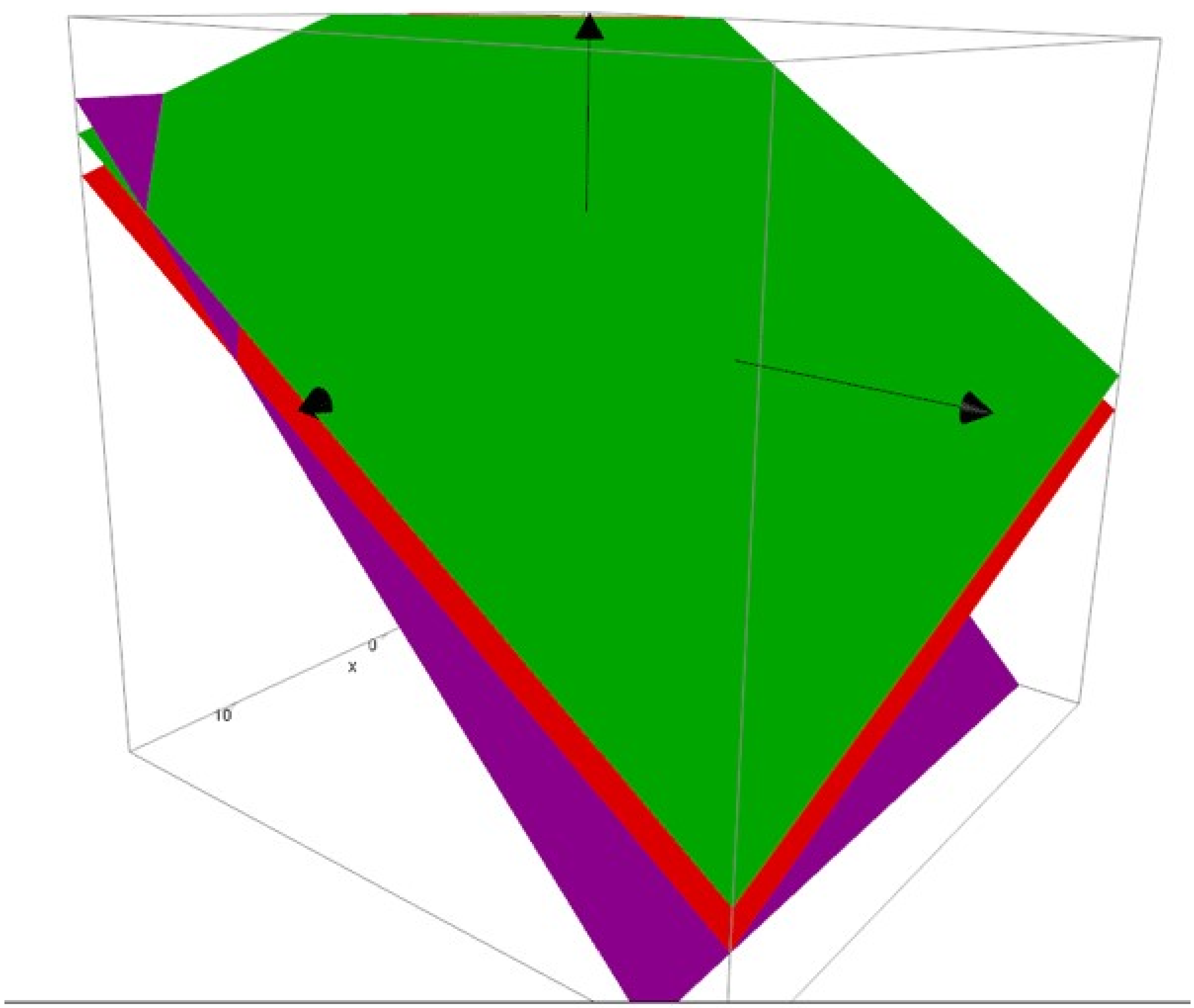
$$t = z = 17 - x = -y - 3$$

Substitute into third equation

$$x + 2y + 3z = 16$$
$$x + 2(-3 - t) + 3t = 16$$
$$x - 6 - 2t + 3t = 16$$
$$x + t = 22$$
$$t = 22 - x$$

so

$$t = z = 22 - x = -y - 3$$



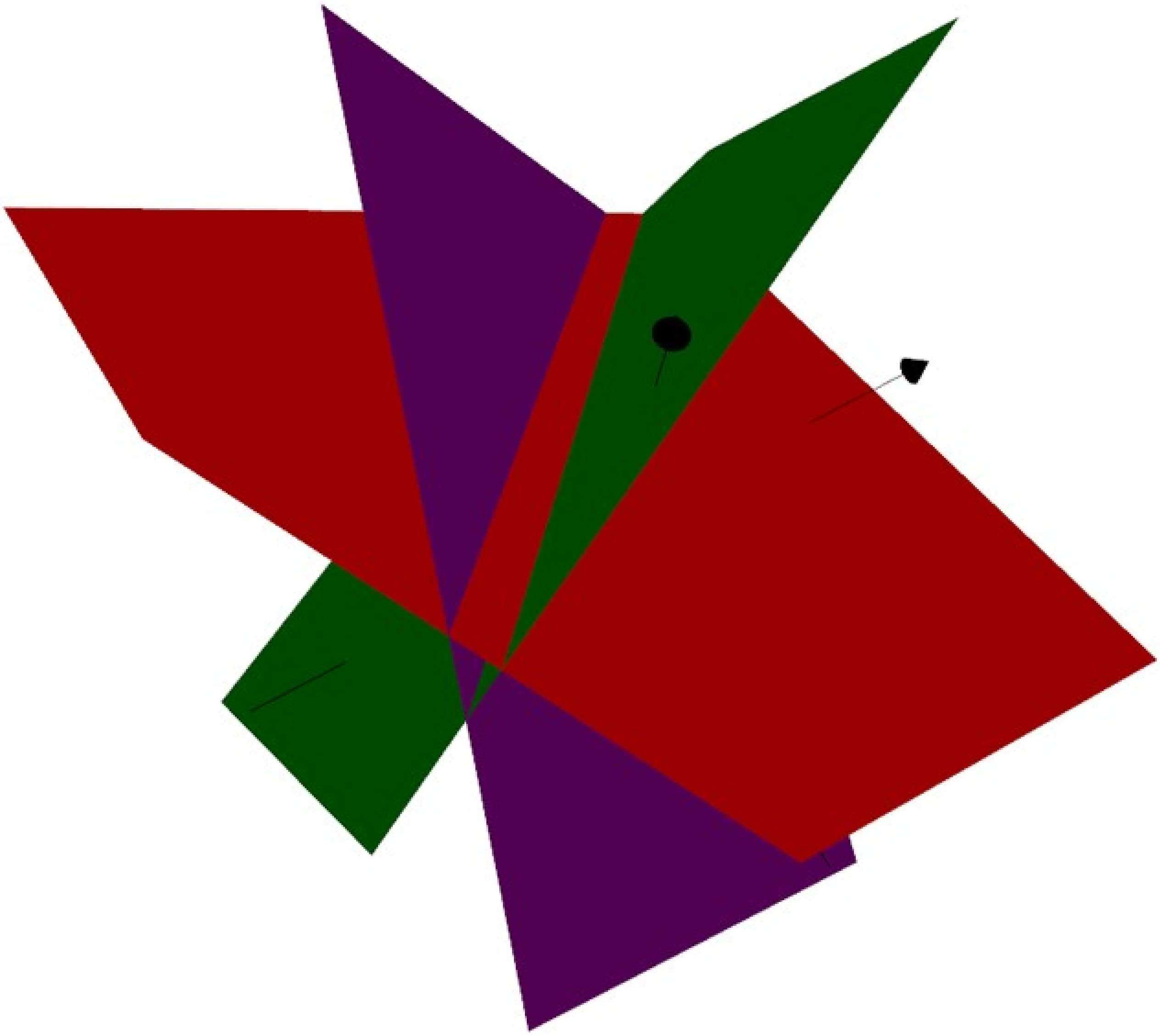
Three lines of intersection
Similar to above.
Examine each pair of planes in turn.

Example

$$3x - y + 2z = 1$$
$$x - 2y - z = -3$$
$$2x + y + 3z = 5$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



A plane of intersection

Two redundant equations

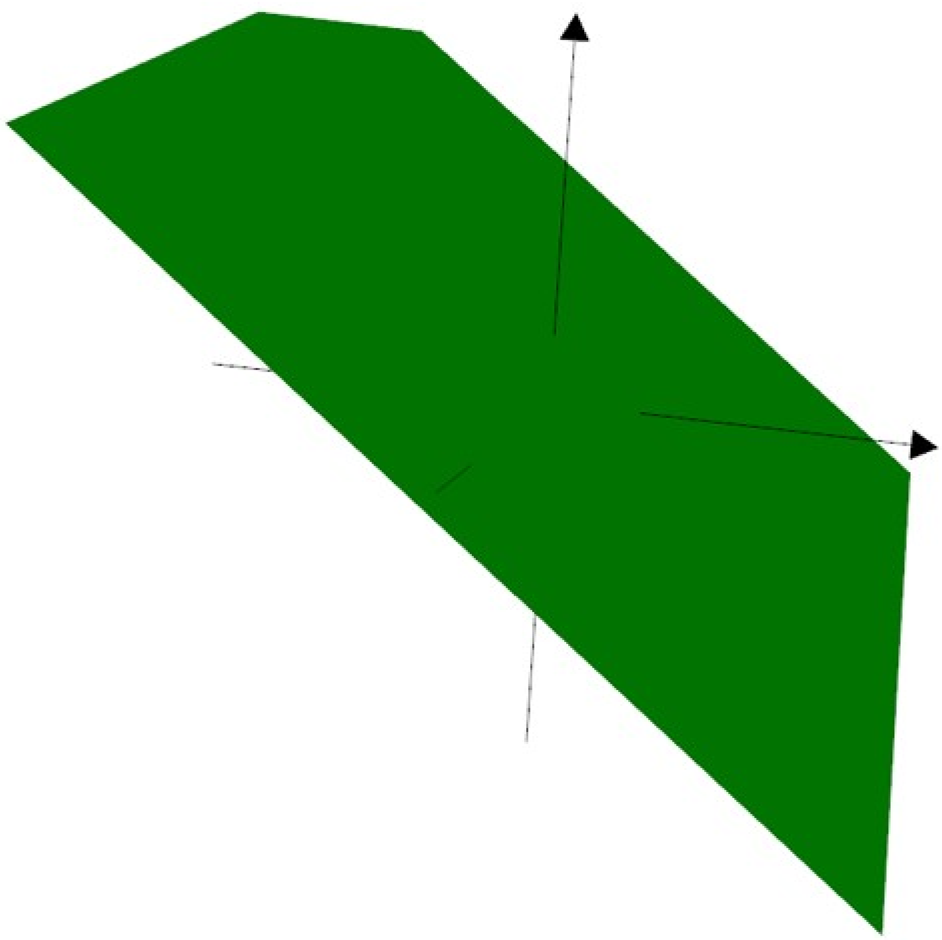
Example

$$3x - y + 4z = 3$$
$$6x - 2y + 8z = 6$$
$$15x - 5y + 20z = 15$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

No consistency



No intersection

Example

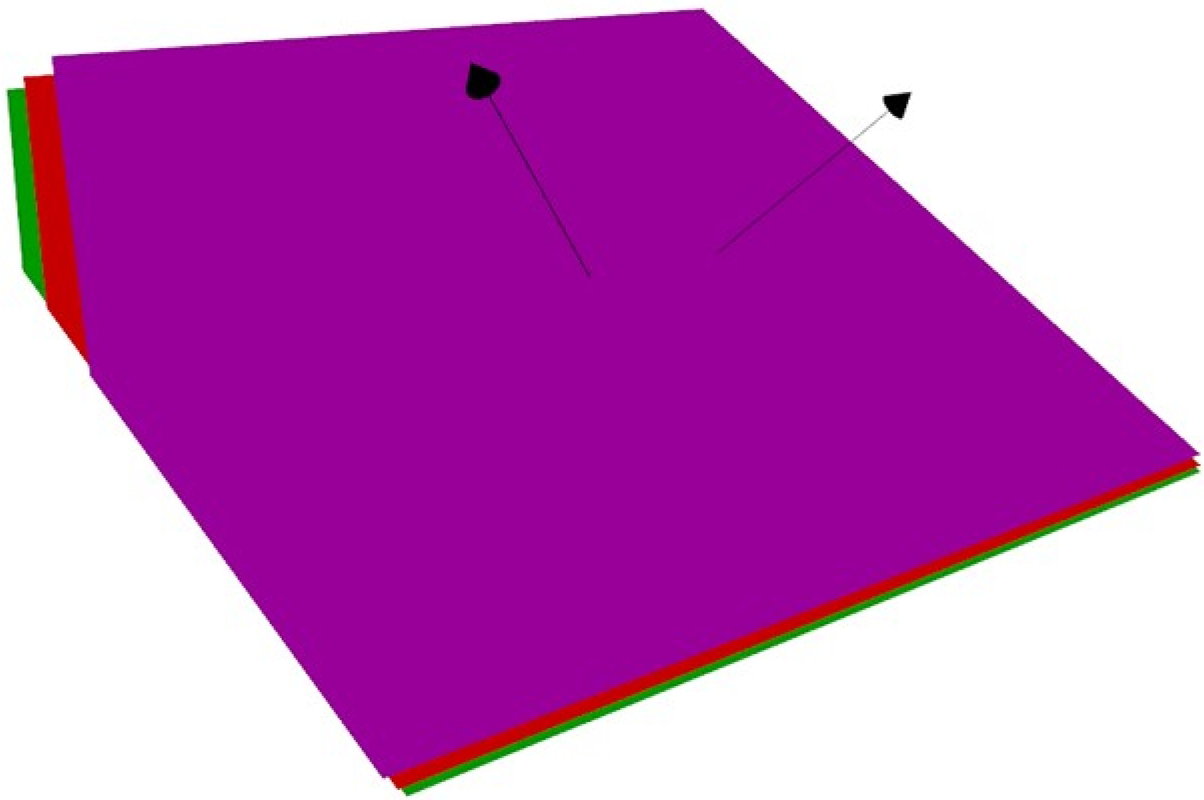
$$3x - y + 4z = 3$$
$$6x - 2y + 8z = 8$$
$$15x - 5y + 20z = 12$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

No consistency

All planes are parallel



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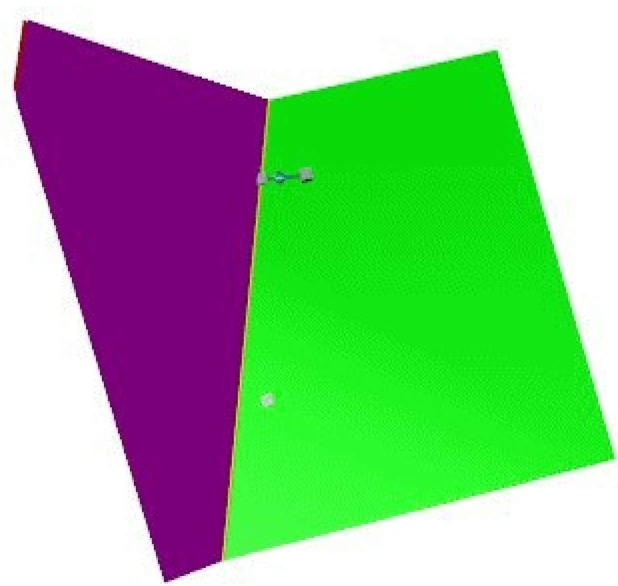
vvvvvvVector Equations

The angle between two planes



The angle between two planes is found
using the scalar product.
 It is equal to the acute angle determined by
the normal vectors of the planes.

Example

Calculate the angle between the planes

π_1: $x + 2y - 2z = 5$

and π_2: $6x - 3y + 2z = 8$

$$\text{let } \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \text{ represent the normal for } \pi_1$$

$$\text{and } \mathbf{b} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} \text{ represent the normal for } \pi_2$$

$$|\mathbf{a}| = \sqrt{1+4+4} \qquad |\mathbf{b}| = \sqrt{36+9+4}$$

$$= 3 \qquad = 7$$

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}||\mathbf{b}|}$$

$$\cos\theta = \frac{1\times 6 - 2\times 3 - 2\times 2}{21}$$

$$\cos\theta = \frac{-4}{21}$$

$$\theta = 100.98^\circ \quad \text{i.e obtuse}$$

$$\theta = 79.02^\circ$$

The distance between parallel planes

Let P be a point on plane π_1 : $ax + by + cz = n$

$\mathbf{a.x} = n$

and Q be a point on plane π_2 : $ax + by + cz = m$

$\mathbf{a.x} = m$

Since the planes are parallel, they share the common normal, **a**

$\mathbf{a}=(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$

The distance between the planes is

$$PQ = \frac{|m-n|}{|\mathbf{a}|}$$

Example

Calculate the distance between the planes

π_1: $x + 2y - 2z = 5$

and π_2: $6x + 12y - 12z = 8$

$$x + 2y - 2z = 5$$

$$6x + 12y - 12z = 8$$

$$x + 2y - 2z = \frac{4}{3}$$

$$so \quad \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad n = 5 \quad \text{and} \quad \mathrm{m} = \frac{4}{3}$$

$$PQ = \frac{|m - n|}{|\mathbf{a}|}$$

$$= \frac{\left|\frac{4}{3} \quad - 5\right|}{\left|\sqrt{1 + 4 + 4}\right|}$$

$$= \frac{\frac{11}{3}}{3}$$

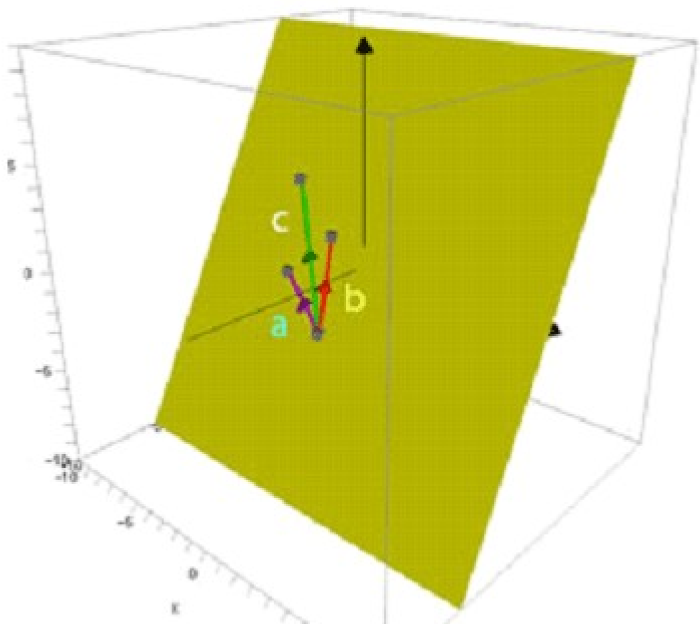
$$= \frac{11}{9}$$

$$= 1\frac{2}{9} \text{ units}$$

Coplanar vectors

If a relationship exists between the vectors **a**, **b** and **c** such that **c=**λ**a+**μ**b, where** λ and μ are constants**,** **then vectors a, b and c are co-planar.**

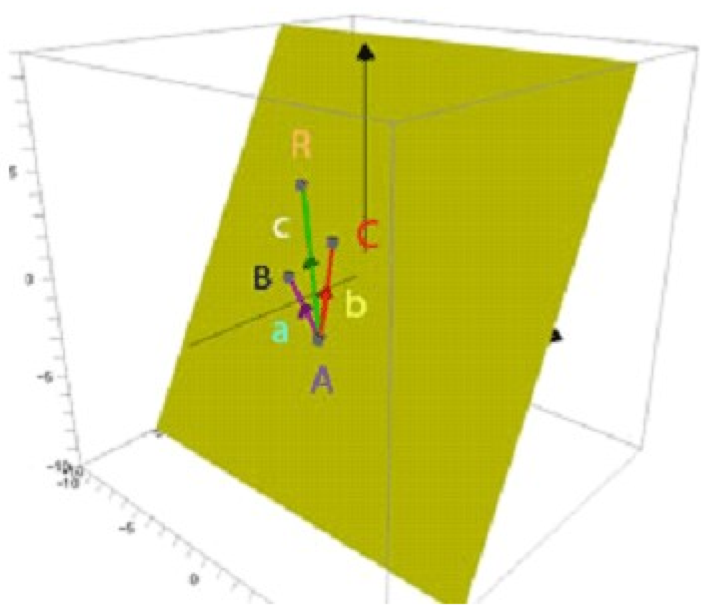
If three vectors are co-planar,
c=λ**a+**μ**b**



Vector equation of a plane

From the coplanar section above,
c=λa+μb

When position vectors are used,



$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$$

$$\overrightarrow{AR} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

$$\mathbf{r} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) + \mathbf{a}$$

$$\mathbf{r} = \lambda\mathbf{b} - \lambda\mathbf{a} + \mu\mathbf{c} - \mu\mathbf{a} + \mathbf{a}$$

$$\mathbf{r} = \mathbf{a} - \lambda\mathbf{a} - \mu\mathbf{a} + \mu\mathbf{c} + \lambda\mathbf{b}$$

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

r=(1-λ-u)**a**+ λ**b+**μ**c** is the **vector equation of the plane.**

Since λ and b are variable, there will be many possible equations for the plane.

Effects of changing λ and μ

Example

Find a vector equation of the plane through the points
A (-1,-2,-3) , B(-2,0,1) and C (-4,-1,-1)

$$\mathbf{r} = (1-\lambda-\mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

$$= (1-\lambda-\mu)\begin{pmatrix}-1\\-2\\-3\end{pmatrix} + \lambda\begin{pmatrix}-2\\0\\1\end{pmatrix} + \mu\begin{pmatrix}-4\\-1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-(1-\lambda-\mu)-2\lambda-4\mu\\-2(1-\lambda-\mu)-\mu\\-3(1-\lambda-\mu)+\lambda-\mu\end{pmatrix}$$

$$= \begin{pmatrix}-1+\lambda+\mu-2\lambda-4\mu\\-2+2\lambda+2\mu-\mu\\-3+3\lambda+3\mu+\lambda-\mu\end{pmatrix}$$

$$= \begin{pmatrix}-1-\lambda-3\mu\\-2+2\lambda+\mu\\-3+4\lambda+2\mu\end{pmatrix}$$

$$= (-1-\lambda-3\mu)\mathbf{i} + (-2+2\lambda+\mu)\mathbf{j} + (-3+4\lambda+2\mu)\mathbf{k}$$

If λ = 2 and μ =3

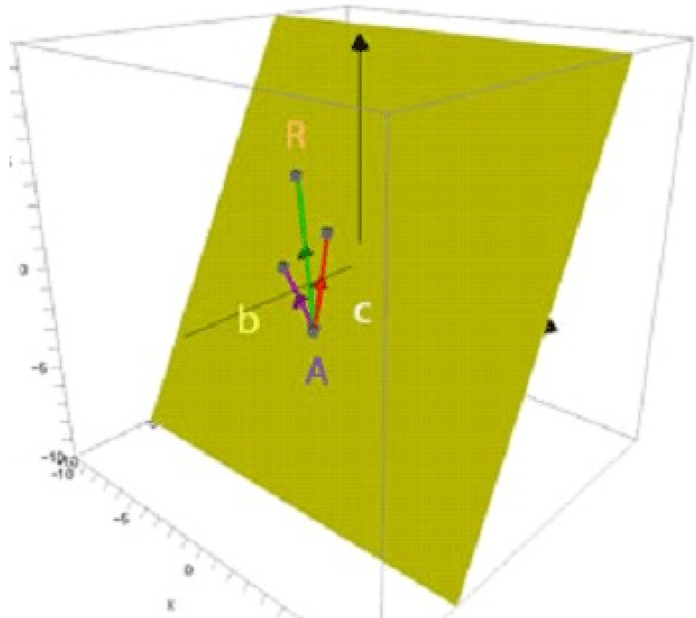
$$\mathbf{r} = (-1-\lambda-3\mu)\mathbf{i} + (-2+2\lambda+\mu)\mathbf{j} + (-3+4\lambda+2\mu)\mathbf{k}$$

$$\mathbf{r} = (-1-2-9)\mathbf{i} + (-2+4+3)\mathbf{j} + (-3+8+6)\mathbf{k}$$

$$\mathbf{r} = -12\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}$$

When A is a known point on the plane,
R is any old point on the plane and **b** and **c** are vectors parallel to the plane,

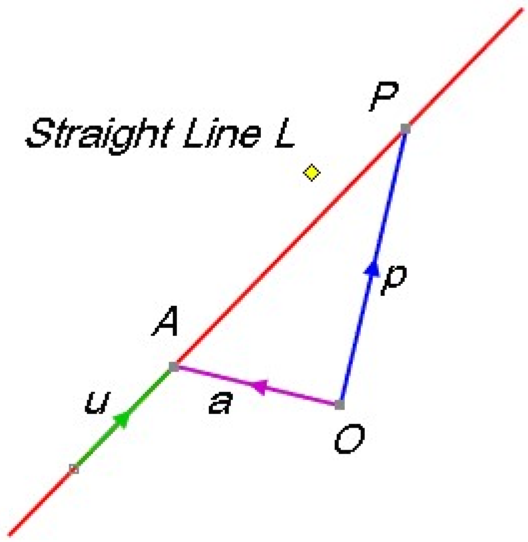
the **vector equation of the plane** is
r=a+λ**b+**μ**c**



The equations of a line

A line can be described when a point on it and its direction vector – a vector parallel to the line – are known.

In the diagram below, the line L passes through points $A(x_1,y_1,z_1)$ and P (x,y,z).



uis the direction vector a**i** +b**j** +c**k**
Being on the line, it has the same direction as any parallel line.

O is the origin.
a and **p** represent the position vectors of A and P.

P is on line L

$\Rightarrow \quad \overrightarrow{AP} = \lambda \mathbf{u}$ for some scalar λ

$\Rightarrow \quad \mathbf{p} - \mathbf{a} = \lambda \mathbf{u}$

$\Rightarrow \quad \mathbf{p} = \mathbf{a} + \lambda \mathbf{u}$

$$\mathbf{p} = \mathbf{a} + \lambda \mathbf{u}$$

is the vector equation of the line

convention often replaces **p** with **r**

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$

If two points are known, say A and B

then $\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} - \lambda\mathbf{a}$

$\Rightarrow \qquad \mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$

In component form, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 + \lambda a \\ y_1 + \lambda b \\ z_1 + \lambda c \end{pmatrix}$$

giving the parametric equations

$$x = x_1 + \lambda a \quad , \qquad y = y_1 + \lambda b \quad , \; z = z_1 + \lambda c$$

so

$$\frac{x - x_1}{a} = \lambda \qquad \frac{y - y_1}{b} = \lambda \qquad \frac{z - z_1}{c} = \lambda$$

Giving the symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

This is also known as :

standard form,

canonical form,

co-ordinate equation

Example

Find the vector equation of the straight line through (3,2,1) which is parallel to the vector 2**i** +3**j** +4**k**

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$$

$$\Rightarrow \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

are the vector equations of the line

Example

Find the vector form of the equation of the straight line which has parametric equations

$$x = 4 - 2\lambda \qquad y = 7 + \lambda \qquad z = 3 - 4\lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{r} = 4\mathbf{i} + 7\mathbf{j} + 3\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$$

Example

Find the Cartesian form of the line which has position vector 3**i** +2**j** +**k** and is parallel to the vector **i** - **j** + **k**

$$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\Rightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore x = 3 + \lambda \qquad y = 2 - \lambda \qquad z = 1 + \lambda$$

$$\frac{x-3}{1} = \frac{y-2}{-1} = \frac{z-1}{1} = \lambda$$

$$\Rightarrow \quad x - 3 = 2 - y = z - 1 = \lambda$$

Example

Find the vector equation of the line passing through A(1,2,3) and B(4,5,6)

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$$

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$\Rightarrow \quad \mathbf{u} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\Rightarrow \qquad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

alternatively

$$\mathbf{r} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$$

$$\Rightarrow \quad \mathbf{r} = (1-\lambda)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\Rightarrow \quad \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\Rightarrow \quad \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

Example

The vector equation of a line is

$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

State the point with z co-ordinate 3 which also lies on this line.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + 2\lambda \qquad y = 2 - \lambda \qquad z = 6 + 3\lambda$$

When $z = 3$

$$3 = 6 + 3\lambda$$

$$\Rightarrow \quad \lambda = \frac{3-6}{3} = -1$$

$$\Rightarrow \quad x = 3 - 2 = 1 \qquad y = 2 + 1 = 3 \qquad z = 6 - 3 = 3$$

$\Rightarrow$ point $(1,3,3)$ lies on line

Example

A line L has equations

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{3-z}{4}$$

Is the vector $\mathbf{s} = 6\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$ parallel to L ?

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{3-z}{4} = \lambda$$

$$\Rightarrow \quad x = -2 + 3\lambda \qquad y = 1 + 2\lambda \qquad z = 3 - 4\lambda$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$$

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$$

(-2,1,3) is a point on L

and $\lambda(3\mathbf{i}+2\mathbf{j}-4\mathbf{k})$ is a direction vector.

$\mathbf{s}$ has direction ratio $6:4:-8=3:2:-4$

The direction ratios of $\mathbf{s}$ and $\mathbf{u}$ are the same

$\Rightarrow \quad \mathbf{s} \parallel \mathbf{u}$

The angle between a line and a plane

The angle θ between a line and a plane is the complement of the angle between the line and the normal to the plane.

If the line has direction vector **u** and the normal to the plane is **a,** then

$$\sin\theta^\circ = \left|\cos(90-\theta)^\circ\right| = \frac{|\mathbf{a}\bullet\mathbf{u}|}{|\mathbf{a}||\mathbf{u}|}$$

Example

Given the equations

$$\frac{x-4}{3}=\frac{y-3}{2}=\frac{z-5}{6}$$

and the plane $6x+3y-2z=14$

1) Find the point of intersection

2) Find the angle the line makes with the plane.

1)

$$\frac{x-4}{3}=\frac{y-3}{2}=\frac{z-5}{6}=\lambda$$

$$\Rightarrow \quad x=4+3\lambda \qquad y=3+2\lambda \qquad z=5+6\lambda$$

$\therefore (\ 4+3\lambda\ ,\ 3+2\lambda\ ,5+6\lambda\ )$ lies on the plane

$$6(4+3\lambda)+3(3+2\lambda\ )-2(5+6\lambda\ )=14$$

$$24+18\lambda+9+6\lambda-10-12\lambda=14$$

$$23+12\lambda=14$$

$$\lambda=\frac{14-23}{12}=\frac{-3}{4}$$

$$x=4+3\times\frac{-3}{4} \qquad y=3+2\times\frac{-3}{4} \qquad z=5+6\times\frac{-3}{4}$$

$$x=\frac{16-9}{4} \qquad y=\frac{12-6}{4} \qquad z=\frac{20-18}{4}$$

$$x=\frac{7}{4} \qquad y=\frac{3}{2} \qquad z=\frac{1}{2}$$

The point of intersection is $\left(\frac{7}{4}, \frac{3}{2}, \frac{1}{2}\right)$

2)

$$\sin\theta^\circ = \left|\cos(90-\theta)^\circ\right| = \frac{|\mathbf{a}\cdot\mathbf{u}|}{|\mathbf{a}||\mathbf{u}|}$$

$$\mathbf{a} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$
$$\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

$$\sin\theta^\circ = \frac{\left|(6\mathbf{i}+3\mathbf{j}-2\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}+6\mathbf{k})\right|}{\left(\left|\sqrt{36+9+4}\right|\right)\left|\left(\sqrt{9+4+36}\right)\right|}$$

$$\Rightarrow \quad \sin\theta^\circ = \frac{12}{49} \qquad (o \le \theta \le 90)$$
$$\Rightarrow \quad \theta = 14.175^\circ$$

The angle of intersection is 14.2°

The intersection of two lines

Example

Show that the lines with equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\frac{x+1}{12} = \frac{y-7}{6} = \frac{z-5}{3} = \lambda_2$$

intersect and find the point of intersection and the equation of the plane containing the lines.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + 4\lambda_1 \qquad y = 4 + \lambda_1 \qquad z = 1$$

and

$$\frac{x+1}{12} = \frac{y-7}{6} = \frac{z-5}{3} = \lambda_2$$

$$\Rightarrow \quad x = -1 + 12\lambda_2 \qquad y = 7 + 6\lambda_2 \qquad z = 5 + 3\lambda_2$$

Equating co-ordinates

$3+4\lambda_1 = -1+12\lambda_2 \qquad 4+\lambda_1 = 7+6\lambda_2 \qquad 1 = 5+3\lambda_2$

$4\lambda_1 = -4+12\lambda_2 \qquad (1)$

$\lambda_1 = 3+6\lambda_2 \qquad (2)$

$0 = 4+3\lambda_2 \qquad (3)$

From (3), $3\lambda_2 = -4$

$\Rightarrow \lambda_2 = \dfrac{-4}{3}$

$\Rightarrow \lambda_1 = 3+6\times\dfrac{-4}{3} = -5$

substituting

$x = 3+4\lambda_1 \qquad y = 4+\lambda_1 \qquad z = 1$

$= 3-20 \qquad = 4-5$

$= -17 \qquad = -1$

Intersection point is (-17, -1,1)

Let A(-17,-1,1) B(3,4,1) C(-1,7,5) be the points from the lines above

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ 4 \end{pmatrix}$$

$$\mathbf{n} \bullet \overrightarrow{\mathrm{AP}} = 0$$

Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{p} = \overrightarrow{OP}$, so $\overrightarrow{\mathrm{AP}} = \overrightarrow{OP} - \overrightarrow{OA}$

$\Rightarrow \quad \mathbf{n} \bullet (\mathbf{p} - \mathbf{a}) = 0$

Here, $\mathbf{n} = \left| \overrightarrow{AB} \times \overrightarrow{AC} \right|$

$$\mathbf{n} = \left| \overrightarrow{AB} \times \overrightarrow{AC} \right| = \left| \begin{pmatrix} i & j & k \\ 20 & 5 & 0 \\ 16 & 8 & 4 \end{pmatrix} \right|$$

$$= 20\mathbf{i} - 80\mathbf{j} + 80\mathbf{k}$$

$$= \mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{p} - \mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x+17 \\ y+1 \\ z-1 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{n} \bullet \overrightarrow{AP} = 0$$

$$\Rightarrow \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} x+17 \\ y+1 \\ z-1 \end{pmatrix} = 0$$

$$\Rightarrow x + 17 - 4(y+1) + 4(z-1) = 0$$

$$\Rightarrow x + 17 - 4y - 4 + 4z - 4 = 0$$

$$\Rightarrow x - 4y + 4z + 9 = 0$$

The intersection of two planes

To find the equations of the line of intersection of two planes, a direction vector and point on the line is required.

Since the line of intersection lies in both planes, the direction vector is parallel to the vector products of the normal of each plane.

Example

Find the equation for the line of intersection of the planes

-3x + 2y + z = -5
7x + 3y - 2z = -2

$-3x+2y+z=-5 \qquad\qquad 7x+3y-2z=-2$

Let $z=0$

Then $-3x+2y=-5.....(1)$

and $7x+3y=-2.....(2)$

$(2)\times 2 \qquad 14x+6y=-4$

$(1)\times -3 \qquad 9x-6y=15$

add $\qquad 23x=11$

$$\Rightarrow \qquad x=\frac{11}{23}$$

subst in (1)

$$-\frac{33}{23}+2y=-5$$

$$\Rightarrow \qquad y=\frac{-5+\frac{33}{23}}{2}=\frac{-41}{23}$$

The point $\left(\frac{11}{23}, \frac{-41}{23}, 0\right)$ is on the line of intersection

Normal vectors are $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

and $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \left| \begin{pmatrix} i & j & k \\ -3 & 2 & 1 \\ 7 & 3 & -2 \end{pmatrix} \right|$$

$$= -7\mathbf{i} + \mathbf{j} - 23\mathbf{k}$$

$$\mathbf{r} = \begin{pmatrix} \frac{11}{23} \\ \frac{-41}{23} \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -7 \\ 1 \\ -23 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{23} \\ \frac{-41}{23} \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} \frac{7}{23} \\ \frac{-1}{23} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad x = \frac{11}{23} + \frac{7}{23}\lambda_1 \qquad y = \frac{-41}{23} - \frac{1}{23}\lambda_1 \qquad z = \lambda_1$$

The distance from a point to a plane

To find the distance of a point P to a plane

1. Find the equation of the projection PP′ by using the normal to the plane and the point P.
2. Find the co-ordinates of P′ , the intersection with the plane.
3. Apply the distance formula to PP′

Alternatively

The distance D between a point $P_0(x_0,y_0,z_0)$ and the plane $ax+by+cz+d=0$ is

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Example

Find the distance between the point (3,1,-2) and the plane x +2y + 2z = - 4

$$\mathbf{r} = u+\lambda_1\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix}3\\1\\-2\end{pmatrix}+\lambda_1\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\Rightarrow \quad x=3+\lambda_1 \qquad y=1+2\lambda_1 \qquad z=-2+2\lambda_1$$

Plane equation is $x+2y+2z+4=0$

$$\Rightarrow 3+\lambda_1 + 2(1+2\lambda_1) + 2(-2+2\lambda_1) + 4 = 0$$

$$\Rightarrow 3+\lambda_1 + 2 + 4\lambda_1 - 4 + 4\lambda_1 + 4 = 0$$

$$\Rightarrow 5+9\lambda_1 = 0$$

$$\Rightarrow \lambda_1 = \frac{-5}{9}$$

$$\Rightarrow \quad x = 3-\frac{5}{9} \qquad y = 1-\frac{10}{9} \qquad z = -2-\frac{10}{9}$$

$$P'\left(\frac{22}{9}, -\frac{1}{9}, -\frac{28}{9}\right)$$

$$PP' = \begin{pmatrix} \frac{-5}{9} \\ -\frac{10}{9} \\ -\frac{10}{9} \end{pmatrix} = \frac{-5}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow |PP'| = \left|\frac{-5}{9}\sqrt{1+4+4}\right|$$

$$= \left|\frac{-5}{3}\right|$$

$$= \frac{5}{3} \text{ units}$$

Alternatively

$$x + 2y + 2z = -4 \qquad \text{at } (3,1,-2)$$

$$\Rightarrow \; x + 2y + 2z + 4 = 0$$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|3 + 2 - 4 + 4|}{\sqrt{1 + 4 + 4}}$$

$$= \frac{5}{3}$$

The distance is $\frac{5}{3}$ units

The distance from a point to a line

To find the distance of a point P to a Line L

1. Let the line have direction vector **u** and parameter λ
2. Find the co-ordinates of PP′ by using the scalar product with **u** and the point P.
3. Apply the distance formula to PP′

Find the distance between the line

$$\frac{x+3}{-6} = \frac{y-2}{9} = \frac{z+8}{6}$$

and the point P $(-1, 7, 4)$

$$P' = \begin{pmatrix} -3 \\ 2 \\ -8 \end{pmatrix} + \lambda_1 \begin{pmatrix} -6 \\ 9 \\ 6 \end{pmatrix}$$

$\Rightarrow \quad x = -3 - 6\lambda_1 \qquad y = 2 + 9\lambda_1 \qquad z = -8 + 6\lambda_1$

$P'(-3 - 6\lambda_1\ ,\ 2+9\lambda_1\ ,-8+6\lambda_1)$

$$\overrightarrow{PP'} = \begin{pmatrix} -3-6\lambda \\ 2+9\lambda_1 \\ -8+6\lambda_1 \end{pmatrix} - \begin{pmatrix} -1 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -2-6\lambda \\ -5+9\lambda \\ -12+6\lambda \end{pmatrix}$$

$\overrightarrow{PP'} \bullet \mathbf{u} = 0$

$$\Rightarrow \quad \begin{pmatrix} -2-6\lambda \\ -5+9\lambda \\ -12+6\lambda \end{pmatrix} \bullet \begin{pmatrix} -6 \\ 9 \\ 6 \end{pmatrix} = 0$$

$\Rightarrow -6(-2-6\lambda) + 9(-5+9\lambda) + 6(-12+6\lambda) = 0$

$\Rightarrow 12 + 36\lambda - 45 + 81\lambda - 72 + 36\lambda = 0$

$\Rightarrow -105 + 153\lambda = 0$

$$\Rightarrow \lambda = \frac{105}{153} = \frac{35}{51}$$

$$\overrightarrow{PP'} = \begin{pmatrix} -2-6\lambda \\ -5+9\lambda \\ -12+6\lambda \end{pmatrix} == \begin{pmatrix} -2-6\times\frac{35}{51} \\ -5+9\times\frac{35}{51} \\ -12+6\times\frac{35}{51} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-104}{17} \\ \frac{20}{17} \\ \frac{-134}{17} \end{pmatrix} == \frac{1}{17}\begin{pmatrix} -104 \\ 20 \\ -134 \end{pmatrix}$$

$$\Rightarrow PP' = \frac{1}{17}\sqrt{29172} = 10.04$$

The distance is 10.04 units

The intersection of three planes

To solve the intersection,
use the equations of the plane ax +by +cz +d = 0
to form an augmented matrix, which is solved
for x, y and z.

The intersection between three planes could be:

A single point

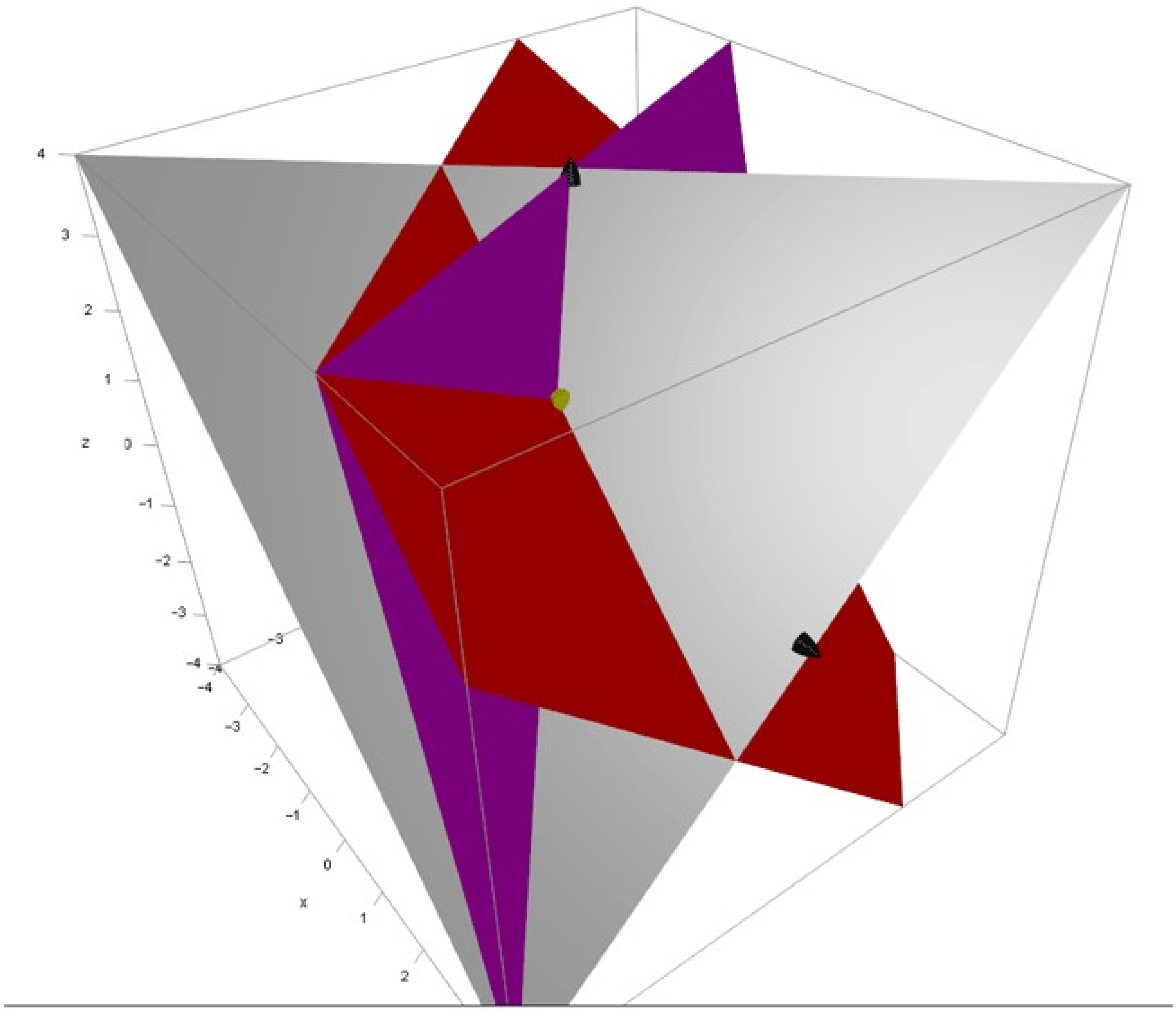
A unique solution is found

Example

$$x + y + z = 2$$
$$4x + 2y + z = 4$$
$$x - y + z = 4$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$Point \quad (1, -1, \ 2)$$



A line of intersection

An infinite number of solutions exist

Example

$$x + 2y + 2z = 11$$
$$x - y + 3z = 8$$
$$4x - y + 11z = 35$$

$$\begin{pmatrix} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

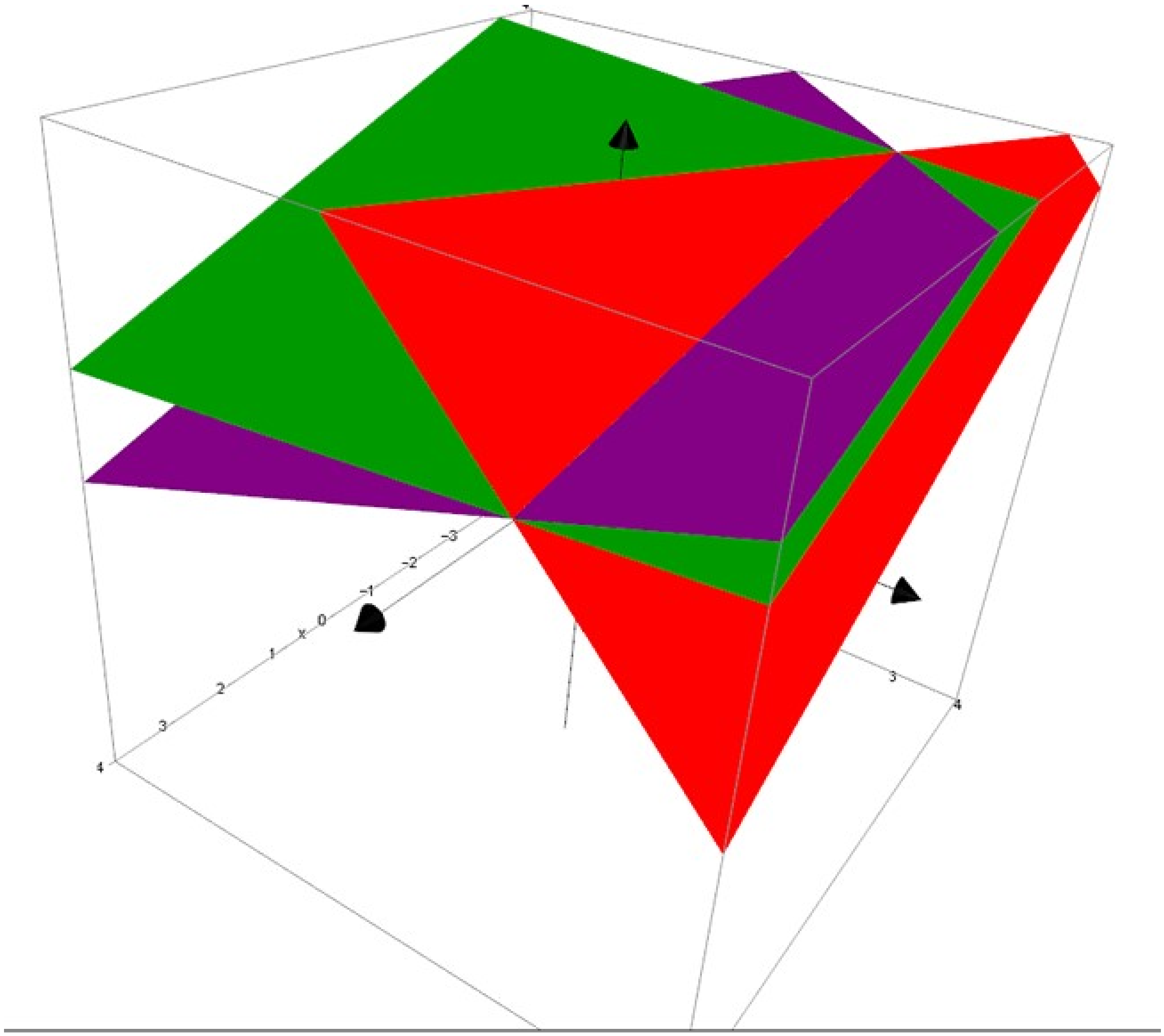
$$x + 2y + 2z = 11$$

$$x = 11 - 2y - 2z$$

$$x = 11 - 2\left(\frac{z+3}{3}\right) - 2z \qquad -3y + z = -3$$

$$x = 13 - \frac{8z}{3} = \frac{39 - 8z}{3} \quad \Rightarrow y = \frac{z+3}{3} \qquad z = z$$

Parametric equations



Two lines of intersection

An infinite number of solutions

Example

$$2x + 4y + 6z = 22$$
$$3y + 3z = -9$$
$$x + 2y + 3z = 16$$

which reduces to

$$\begin{pmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

The system is inconsistent

Using the second row

$$\text{let } z = t$$

$$so$$

$$y + t = -3$$

$$y = -3 - t$$

Substitute into first row

$$x + 2y + 3z = 11$$

$$x + 2(-3 - t) + 3t = 11$$

$$x - 6 - 2t + 3t = 11$$

$$x + t = 17$$

$$x = 17 - t$$

$$so$$

$$t = z = 17 - x = -y - 3$$

Substitute into third equation

$$x + 2y + 3z = 16$$

$$x + 2(-3 - t) + 3t = 16$$

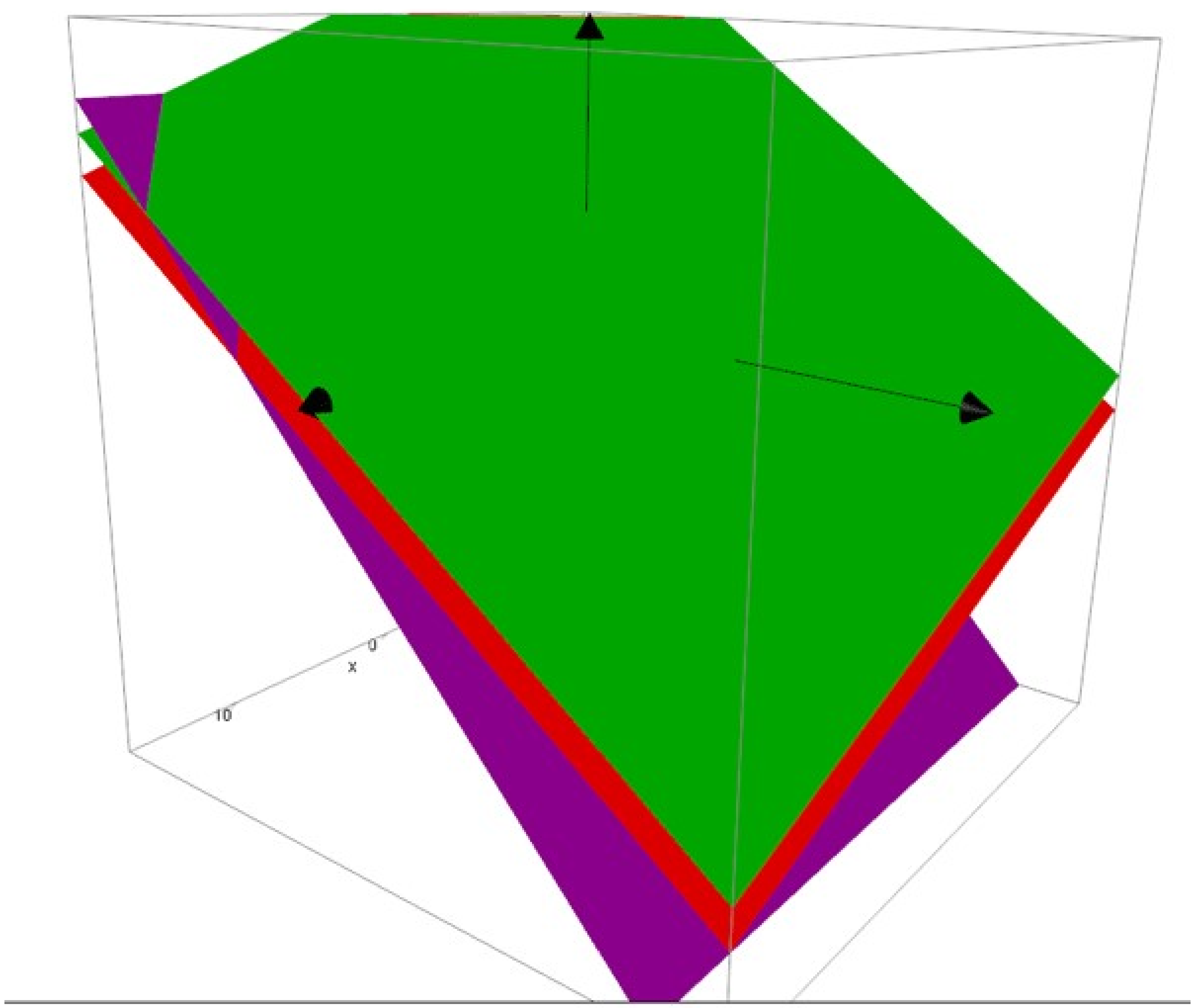
$$x - 6 - 2t + 3t = 16$$

$$x + t = 22$$

$$t = 22 - x$$

so

$$t = z = 22 - x = -y - 3$$



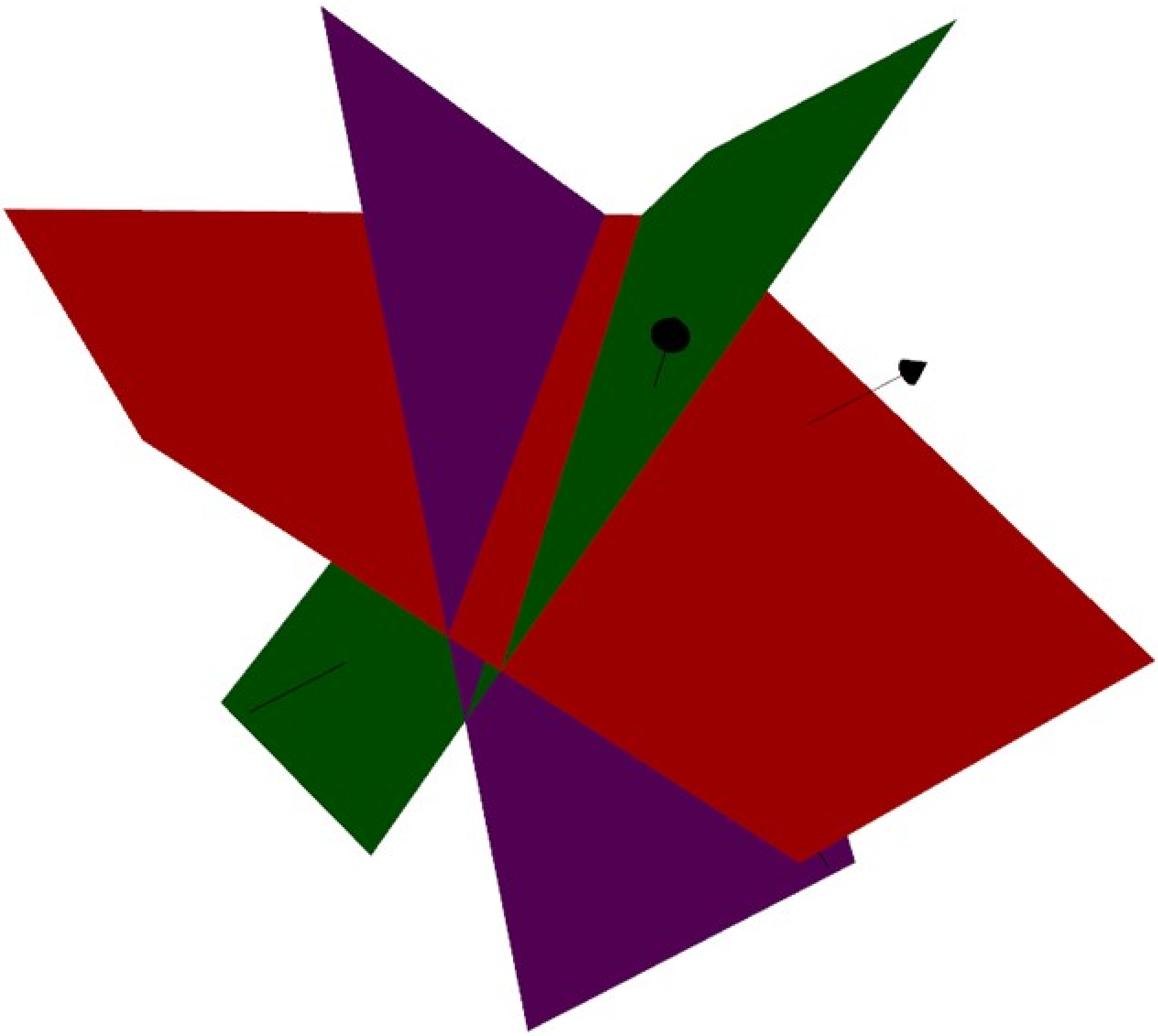
Three lines of intersection
Similar to above.
Examine each pair of planes in turn.

Example

$$3x - y + 2z = 1$$
$$x - 2y - z = -3$$
$$2x + y + 3z = 5$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



A plane of intersection

Two redundant equations

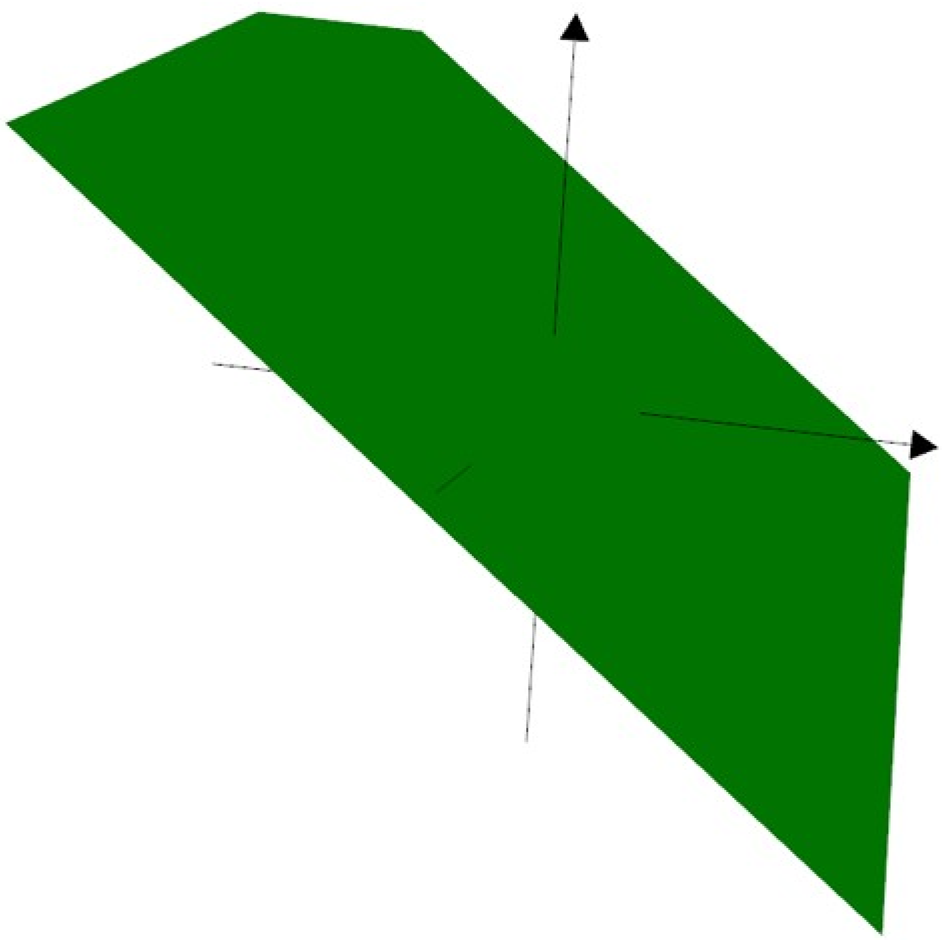
Example

$$3x - y + 4z = 3$$
$$6x - 2y + 8z = 6$$
$$15x - 5y + 20z = 15$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

No consistency



No intersection

Example

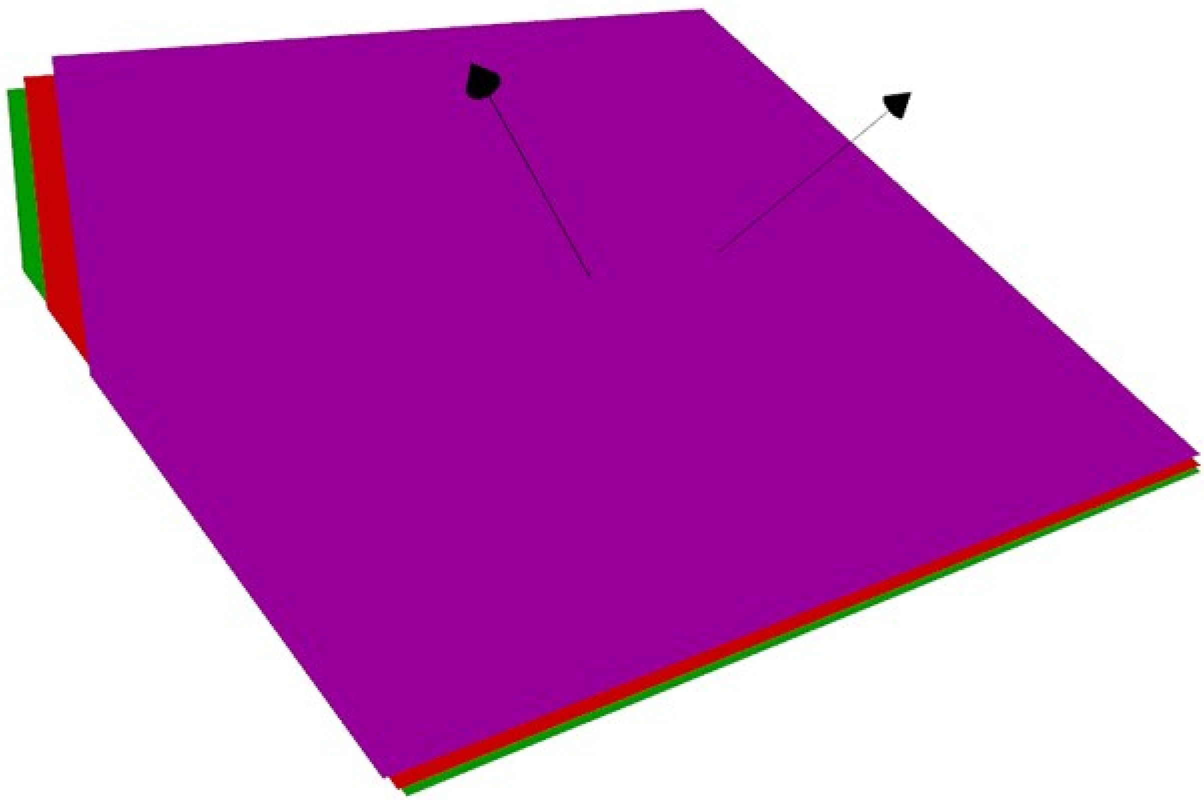
$$3x - y + 4z = 3$$
$$6x - 2y + 8z = 8$$
$$15x - 5y + 20z = 12$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

No consistency

All planes are parallel



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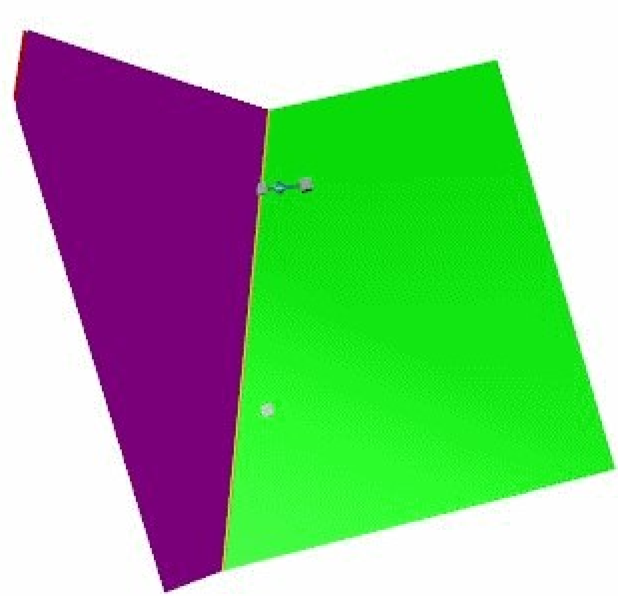
Vector Equations

The angle between two planes



The angle between two planes is found using the scalar product.
It is equal to the acute angle determined by the normal vectors of the planes.

Example

Calculate the angle between the planes

π_1: $x + 2y - 2z = 5$

and π_2: $6x - 3y + 2z = 8$

$$\text{let } \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \text{ represent the normal for } \pi_1$$

$$\text{and } \quad \mathbf{b} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} \text{ represent the normal for } \pi_2$$

$$|\mathbf{a}| = \sqrt{1+4+4} \qquad |\mathbf{b}| = \sqrt{36+9+4}$$

$$= 3 \qquad\qquad = 7$$

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}||\mathbf{b}|}$$

$$\cos\theta = \frac{1\times 6 - 2\times 3 - 2\times 2}{21}$$

$$\cos\theta = \frac{-4}{21}$$

$$\theta = 100.98^\circ \quad \text{i.e obtuse}$$

$$\theta = 79.02^\circ$$

The distance between parallel planes

Let P be a point on plane π_1 : $ax + by + cz = n$

$\mathbf{a.x} = n$

and Q be a point on plane π_2 : $ax + by + cz = m$

$\mathbf{a.x} = m$

Since the planes are parallel, they share the common normal, **a**

a=(a**i** +b**j**+c**k**)

The distance between the planes is

$$PQ = \frac{|m-n|}{|\mathbf{a}|}$$

Example

Calculate the distance between the planes

π_1: $x + 2y - 2z = 5$

and π_2: $6x + 12y - 12z = 8$

$$x + 2y - 2z = 5$$

$$6x + 12y - 12z = 8$$

$$x + 2y - 2z = \frac{4}{3}$$

$$so \quad \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad n = 5 \quad \text{and} \quad \text{m} = \frac{4}{3}$$

$$PQ = \frac{|m-n|}{|\mathbf{a}|}$$

$$= \frac{\left|\frac{4}{3} \quad -5\right|}{\left|\sqrt{1+4+4}\right|}$$

$$= \frac{\frac{11}{3}}{3}$$

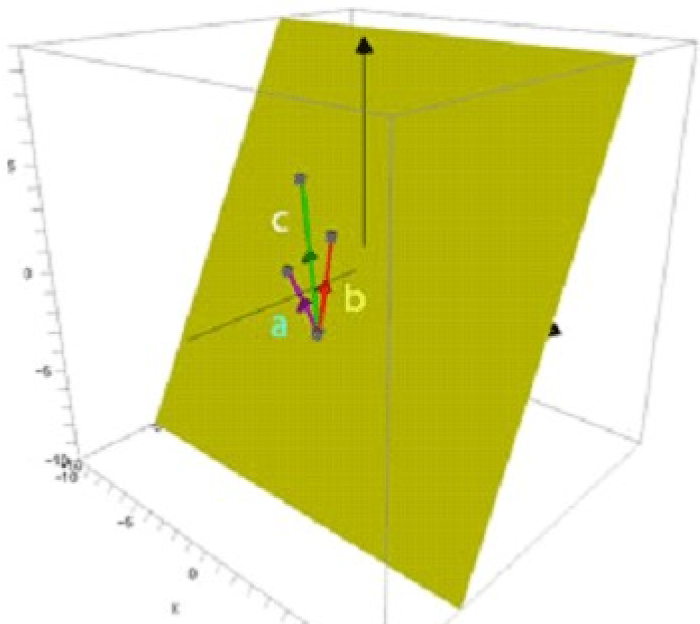
$$= \frac{11}{9}$$

$$= 1\frac{2}{9} \text{ units}$$

Coplanar vectors

If a relationship exists between the vectors **a**, **b** and **c** such that **c=**λ**a+**μ**b, where** λ and μ are constants**,**
then vectors a, b and c are co-planar.

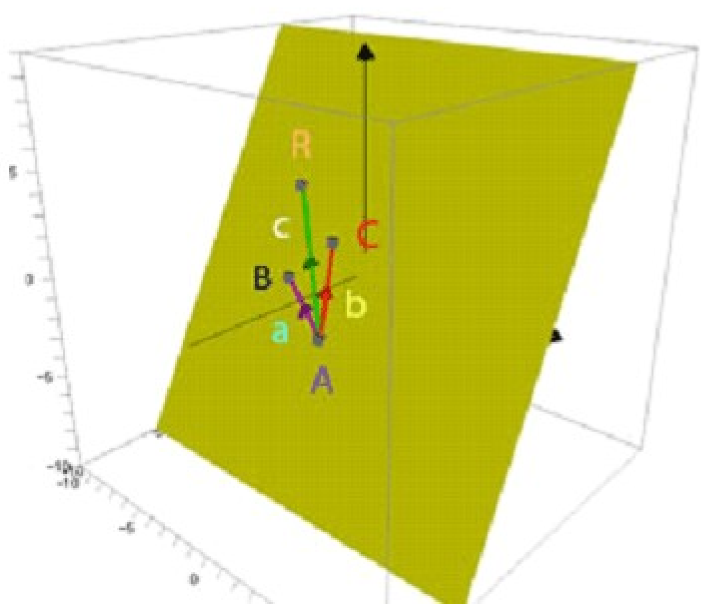
If three vectors are co-planar,
c=λ**a+**μ**b**



Vector equation of a plane

From the coplanar section above,
c=λa+μb

When position vectors are used,



$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$$

$$\overrightarrow{AR} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu\left(\mathbf{c} - \mathbf{a}\right)$$

$$\mathbf{r} = \lambda(\mathbf{b} - \mathbf{a}) + \mu\left(\mathbf{c} - \mathbf{a}\right) + \mathbf{a}$$

$$\mathbf{r} = \lambda\mathbf{b} - \lambda\mathbf{a} + \mu\mathbf{c} - \mu\mathbf{a} + \mathbf{a}$$

$$\mathbf{r} = \mathbf{a} - \lambda\mathbf{a} - \mu\mathbf{a} + \mu\mathbf{c} + \lambda\mathbf{b}$$

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

r=(1-λ-u)**a**+ λ**b+**μ**c** is the **vector equation of the plane.**

Since λ and b are variable, there will be many possible equations for the plane.

Effects of changing λ and μ

Example

Find a vector equation of the plane through the points
A (-1,-2,-3) , B(-2,0,1) and C (-4,-1,-1)

$$\mathbf{r} = (1-\lambda-\mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

$$= (1-\lambda-\mu)\begin{pmatrix}-1\\-2\\-3\end{pmatrix} + \lambda\begin{pmatrix}-2\\0\\1\end{pmatrix} + \mu\begin{pmatrix}-4\\-1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-(1-\lambda-\mu)-2\lambda-4\mu\\-2(1-\lambda-\mu)-\mu\\-3(1-\lambda-\mu)+\lambda-\mu\end{pmatrix}$$

$$= \begin{pmatrix}-1+\lambda+\mu-2\lambda-4\mu\\-2+2\lambda+2\mu-\mu\\-3+3\lambda+3\mu+\lambda-\mu\end{pmatrix}$$

$$= \begin{pmatrix}-1-\lambda-3\mu\\-2+2\lambda+\mu\\-3+4\lambda+2\mu\end{pmatrix}$$

$$= \left(-1-\lambda-3\mu\right)\mathbf{i} + \left(-2+2\lambda+\mu\right)\mathbf{j} + \left(-3+4\lambda+2\mu\right)\mathbf{k}$$

If λ = 2 and μ =3

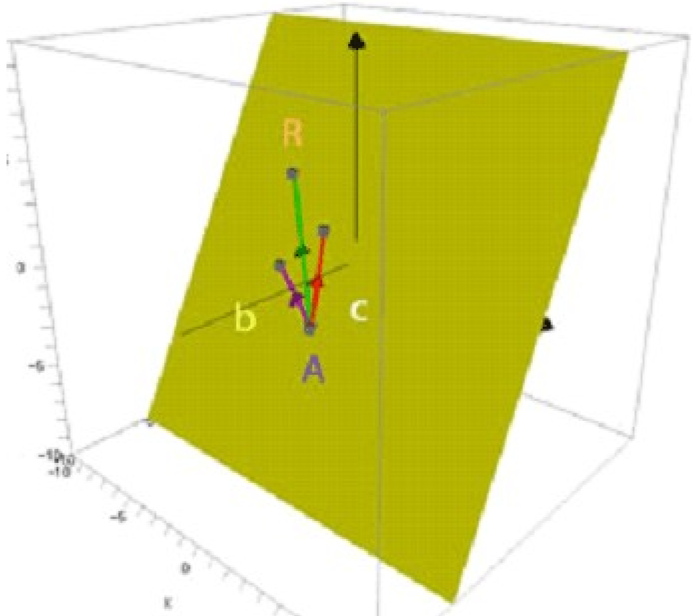
$$\mathbf{r} = \left(-1-\lambda-3\mu\right)\mathbf{i} + \left(-2+2\lambda+\mu\right)\mathbf{j} + \left(-3+4\lambda+2\mu\right)\mathbf{k}$$

$$\mathbf{r} = \left(-1-2-9\right)\mathbf{i} + \left(-2+4+3\right)\mathbf{j} + \left(-3+8+6\right)\mathbf{k}$$

$$\mathbf{r} = -12\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}$$

When A is a known point on the plane,
R is any old point on the plane and **b** and **c** are vectors parallel to the plane,

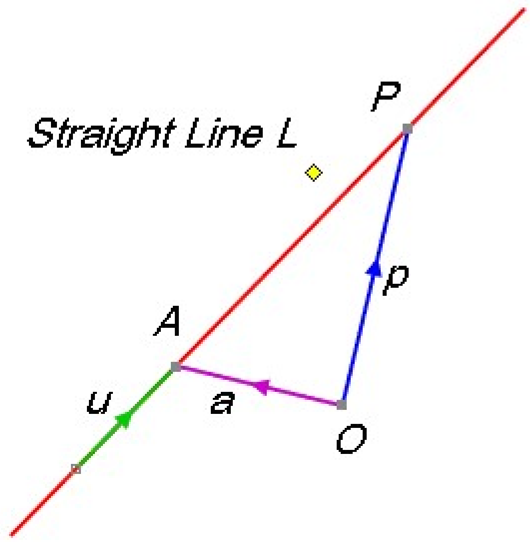
the **vector equation of the plane** is
r=a+λb+μc



The equations of a line

A line can be described when a point on it and its direction vector – a vector parallel to the line – are known.

In the diagram below, the line L passes through points $A(x_1,y_1,z_1)$ and P (x,y,z).



u is the direction vector a**i** +b**j** +c**k**
Being on the line, it has the same direction as any parallel line.

O is the origin.
a and **p** represent the position vectors of A and P.

P is on line L

$\Rightarrow \quad \overrightarrow{AP} = \lambda \mathbf{u}$ for some scalar λ

$\Rightarrow \quad \mathbf{p} - \mathbf{a} = \lambda \mathbf{u}$

$\Rightarrow \quad \mathbf{p} = \mathbf{a} + \lambda \mathbf{u}$

$$\mathbf{p} = \mathbf{a} + \lambda \mathbf{u}$$

is the vector equation of the line

convention often replaces **p** with **r**

$$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$$

If two points are known, say A and B

then $\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} - \lambda\mathbf{a}$

$\Rightarrow \qquad \mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$

In component form, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 + \lambda a \\ y_1 + \lambda b \\ z_1 + \lambda c \end{pmatrix}$$

giving the parametric equations

$$x = x_1 + \lambda a \quad , \qquad y = y_1 + \lambda b \quad , \; z = z_1 + \lambda c$$

so

$$\frac{x - x_1}{a} = \lambda \qquad \frac{y - y_1}{b} = \lambda \qquad \frac{z - z_1}{c} = \lambda$$

Giving the symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

This is also known as :

standard form,

canonical form,

co-ordinate equation

Example

Find the vector equation of the straight line through (3,2,1) which is parallel to the vector 2**i** +3**j** +4**k**

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$$

$$\Rightarrow \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

are the vector equations of the line

Example

Find the vector form of the equation of the straight line which has parametric equations

$$x = 4 - 2\lambda \qquad y = 7 + \lambda \qquad z = 3 - 4\lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

$$\Rightarrow \ \mathbf{r} = 4\mathbf{i} + 7\mathbf{j} + 3\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$$

Example

Find the Cartesian form of the line which has position vector 3**i** +2**j** +**k** and is parallel to the vector **i** - **j** + **k**

$$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\Rightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore x = 3 + \lambda \qquad y = 2 - \lambda \qquad z = 1 + \lambda$$

$$\frac{x-3}{1} = \frac{y-2}{-1} = \frac{z-1}{1} = \lambda$$

$$\Rightarrow \quad x - 3 = 2 - y = z - 1 = \lambda$$

Example

Find the vector equation of the line passing through A(1,2,3) and B(4,5,6)

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$$

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$\Rightarrow \quad \mathbf{u} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\Rightarrow \qquad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

alternatively

$$\mathbf{r} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$$

$$\Rightarrow \mathbf{r} = (1-\lambda)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\Rightarrow \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\Rightarrow \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

Example

The vector equation of a line is

$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

State the point with z co-ordinate 3 which also lies on this line.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + 2\lambda \qquad y = 2 - \lambda \qquad z = 6 + 3\lambda$$

When $z = 3$

$$3 = 6 + 3\lambda$$

$$\Rightarrow \quad \lambda = \frac{3-6}{3} = -1$$

$$\Rightarrow \quad x = 3 - 2 = 1 \qquad y = 2 + 1 = 3 \qquad z = 6 - 3 = 3$$

$\Rightarrow$ point (1,3,3) lies on line

Example

A line L has equations

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{3-z}{4}$$

Is the vector $\mathbf{s} = 6\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$ parallel to L ?

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{3-z}{4} = \lambda$$

$$\Rightarrow \quad x = -2 + 3\lambda \qquad y = 1 + 2\lambda \qquad z = 3 - 4\lambda$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$$

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$$

(-2,1,3) is a point on L

and $\lambda(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ is a direction vector.

$\mathbf{s}$ has direction ratio $6 : 4 : -8 = 3 : 2 : -4$

The direction ratios of $\mathbf{s}$ and $\mathbf{u}$ are the same

$\Rightarrow \quad \mathbf{s} \parallel \mathbf{u}$

The angle between a line and a plane

The angle θ between a line and a plane is the complement of the angle between the line and the normal to the plane.

If the line has direction vector **u** and the normal to the plane is **a,** then

$$\sin\theta^\circ = \left|\cos(90 - \theta)^\circ\right| = \frac{|\mathbf{a} \bullet \mathbf{u}|}{|\mathbf{a}||\mathbf{u}|}$$

Example

Given the equations

$$\frac{x-4}{3}=\frac{y-3}{2}=\frac{z-5}{6}$$

and the plane $6x+3y-2z=14$

1) Find the point of intersection

2) Find the angle the line makes with the plane.

1)

$$\frac{x-4}{3}=\frac{y-3}{2}=\frac{z-5}{6}=\lambda$$

$$\Rightarrow \quad x=4+3\lambda \qquad y=3+2\lambda \qquad z=5+6\lambda$$

$\therefore (\ 4+3\lambda\ ,\ 3+2\lambda\ ,5+6\lambda\ )$ lies on the plane

$$6(4+3\lambda)+3(3+2\lambda\ )-2(5+6\lambda\ )=14$$

$$24+18\lambda+9+6\lambda-10-12\lambda=14$$

$$23+12\lambda=14$$

$$\lambda=\frac{14-23}{12}=\frac{-3}{4}$$

$$x=4+3\times\frac{-3}{4} \qquad y=3+2\times\frac{-3}{4} \qquad z=5+6\times\frac{-3}{4}$$

$$x=\frac{16-9}{4} \qquad y=\frac{12-6}{4} \qquad z=\frac{20-18}{4}$$

$$x=\frac{7}{4} \qquad y=\frac{3}{2} \qquad z=\frac{1}{2}$$

The point of intersection is $\left(\frac{7}{4}, \frac{3}{2}, \frac{1}{2}\right)$

2)

$$\sin\theta° = \left|\cos(90-\theta)°\right| = \frac{|\mathbf{a}\cdot\mathbf{u}|}{|\mathbf{a}||\mathbf{u}|}$$

$$\mathbf{a} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$
$$\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

$$\sin\theta° = \frac{\left|(6\mathbf{i}+3\mathbf{j}-2\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}+6\mathbf{k})\right|}{\left(\left|\sqrt{36+9+4}\right|\right)\left|\left(\sqrt{9+4+36}\right)\right|}$$

$$\Rightarrow \quad \sin\theta° = \frac{12}{49} \qquad (o \le \theta \le 90)$$
$$\Rightarrow \quad \theta = 14.175°$$

The angle of intersection is $14.2°$

The intersection of two lines

Example

Show that the lines with equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\frac{x+1}{12} = \frac{y-7}{6} = \frac{z-5}{3} = \lambda_2$$

intersect and find the point of intersection and the equation of the plane containing the lines.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + 4\lambda_1 \qquad y = 4 + \lambda_1 \qquad z = 1$$

and

$$\frac{x+1}{12} = \frac{y-7}{6} = \frac{z-5}{3} = \lambda_2$$

$$\Rightarrow \quad x = -1 + 12\lambda_2 \qquad y = 7 + 6\lambda_2 \qquad z = 5 + 3\lambda_2$$

Equating co-ordinates

$3+4\lambda_1 = -1+12\lambda_2 \qquad 4+\lambda_1 = 7+6\lambda_2 \qquad 1 = 5+3\lambda_2$

$4\lambda_1 = -4+12\lambda_2 \qquad (1)$

$\lambda_1 = 3+6\lambda_2 \qquad (2)$

$0 = 4+3\lambda_2 \qquad (3)$

From (3), $\quad 3\lambda_2 = -4$

$\Rightarrow \lambda_2 = \dfrac{-4}{3}$

$\Rightarrow \lambda_1 = 3+6\times\dfrac{-4}{3} = -5$

substituting

$$x = 3+4\lambda_1 \qquad y = 4+\lambda_1 \qquad z = 1$$

$$= 3-20 \qquad = 4-5$$

$$= -17 \qquad = -1$$

Intersection point is (-17, -1,1)

Let A(-17,-1,1) B(3,4,1) C(-1,7,5) be the points from the lines above

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ 4 \end{pmatrix}$$

$$\mathbf{n} \bullet \overrightarrow{\mathrm{AP}} = 0$$

Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{p} = \overrightarrow{OP}$, so $\overrightarrow{\mathrm{AP}} = \overrightarrow{OP} - \overrightarrow{OA}$

$$\Rightarrow \quad \mathbf{n} \bullet (\mathbf{p} - \mathbf{a}) = 0$$

Here, $\mathbf{n} = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$\mathbf{n} = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \left|\begin{pmatrix} i & j & k \\ 20 & 5 & 0 \\ 16 & 8 & 4 \end{pmatrix}\right|$$

$$= 20\mathbf{i} - 80\mathbf{j} + 80\mathbf{k}$$

$$= \mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{p}-\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x+17 \\ y+1 \\ z-1 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{n} \bullet \overrightarrow{AP} = 0$$

$$\Rightarrow \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} x+17 \\ y+1 \\ z-1 \end{pmatrix} = 0$$

$$\Rightarrow x + 17 - 4(y+1) + 4(z-1) = 0$$

$$\Rightarrow x + 17 - 4y - 4 + 4z - 4 = 0$$

$$\Rightarrow x - 4y + 4z + 9 = 0$$

The intersection of two planes

To find the equations of the line of intersection of two planes, a direction vector and point on the line is required.

Since the line of intersection lies in both planes, the direction vector is parallel to the vector products of the normal of each plane.

Example

Find the equation for the line of intersection of the planes

-3x + 2y + z = -5
7x + 3y - 2z = -2

$-3x+2y+z=-5 \qquad\qquad 7x+3y-2z=-2$

Let $z=0$

Then $-3x+2y=-5.....(1)$

and $7x+3y=-2.....(2)$

$(2)\times 2 \qquad 14x+6y=-4$

$(1)\times -3 \qquad 9x-6y=15$

add $\qquad 23x=11$

$$\Rightarrow \qquad x=\frac{11}{23}$$

subst in (1)

$$-\frac{33}{23}+2y=-5$$

$$\Rightarrow \qquad y=\frac{-5+\frac{33}{23}}{2}=\frac{-41}{23}$$

The point $\left(\frac{11}{23}, \frac{-41}{23}, 0\right)$ is on the line of intersection

Normal vectors are $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

and $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \left| \begin{pmatrix} i & j & k \\ -3 & 2 & 1 \\ 7 & 3 & -2 \end{pmatrix} \right|$$

$$= -7\mathbf{i} + \mathbf{j} - 23\mathbf{k}$$

$$\mathbf{r} = \begin{pmatrix} \frac{11}{23} \\ \frac{-41}{23} \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -7 \\ 1 \\ -23 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{23} \\ \frac{-41}{23} \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} \frac{7}{23} \\ \frac{-1}{23} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad x = \frac{11}{23} + \frac{7}{23}\lambda_1 \qquad y = \frac{-41}{23} - \frac{1}{23}\lambda_1 \qquad z = \lambda_1$$

The distance from a point to a plane

To find the distance of a point P to a plane

1. Find the equation of the projection PP′ by using the normal to the plane and the point P.
2. Find the co-ordinates of P′ , the intersection with the plane.
3. Apply the distance formula to PP′

Alternatively

The distance D between a point $P_0(x_0,y_0,z_0)$ and the plane $ax+by+cz+d=0$ is

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Example

Find the distance between the point (3,1,-2) and the plane x +2y + 2z = - 4

$$\mathbf{r}=u+\lambda_1\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\mathbf{r}=\begin{pmatrix}3\\1\\-2\end{pmatrix}+\lambda_1\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\Rightarrow \quad x=3+\lambda_1 \qquad y=1+2\lambda_1 \qquad z=-2+2\lambda_1$$

Plane equation is $x+2y+2z+4=0$

$$\Rightarrow 3+\lambda_1+2(1+2\lambda_1)+2(-2+2\lambda_1)+4=0$$

$$\Rightarrow 3+\lambda_1+2+4\lambda_1-4+4\lambda_1+4=0$$

$$\Rightarrow 5+9\lambda_1=0$$

$$\Rightarrow \lambda_1=\frac{-5}{9}$$

$$\Rightarrow \quad x=3-\frac{5}{9} \qquad y=1-\frac{10}{9} \qquad z=-2-\frac{10}{9}$$

$$P'\left(\frac{22}{9},-\frac{1}{9},-\frac{28}{9}\right)$$

$$PP'=\begin{pmatrix}\frac{-5}{9}\\ -\frac{10}{9}\\ -\frac{10}{9}\end{pmatrix}=\frac{-5}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\Rightarrow |PP'|=\left|\frac{-5}{9}\sqrt{1+4+4}\right|$$

$$=\left|\frac{-5}{3}\right|$$

$$=\frac{5}{3}\text{ units}$$

Alternatively

$$x + 2y + 2z = -4 \qquad \text{at } (3,1,-2)$$

$$\Rightarrow \; x + 2y + 2z + 4 = 0$$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|3 + 2 - 4 + 4|}{\sqrt{1 + 4 + 4}}$$

$$= \frac{5}{3}$$

The distance is $\frac{5}{3}$ units

The distance from a point to a line

To find the distance of a point P to a Line L

1. Let the line have direction vector **u** and parameter λ
2. Find the co-ordinates of PP′ by using the scalar product with **u** and the point P.
3. Apply the distance formula to PP′

Find the distance between the line

$$\frac{x+3}{-6} = \frac{y-2}{9} = \frac{z+8}{6}$$

and the point P $(-1, 7, 4)$

$$P' = \begin{pmatrix} -3 \\ 2 \\ -8 \end{pmatrix} + \lambda_1 \begin{pmatrix} -6 \\ 9 \\ 6 \end{pmatrix}$$

$\Rightarrow \quad x = -3 - 6\lambda_1 \qquad y = 2 + 9\lambda_1 \qquad z = -8 + 6\lambda_1$

$P'(-3 - 6\lambda_1 \text{ , } 2+9\lambda_1 \text{ ,} -8+6\lambda_1)$

$$\overrightarrow{PP'} = \begin{pmatrix} -3 - 6\lambda \\ 2 + 9\lambda_1 \\ -8 + 6\lambda_1 \end{pmatrix} - \begin{pmatrix} -1 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 - 6\lambda \\ -5 + 9\lambda \\ -12 + 6\lambda \end{pmatrix}$$

$\overrightarrow{PP'} \bullet \mathbf{u} = 0$

$$\Rightarrow \quad \begin{pmatrix} -2 - 6\lambda \\ -5 + 9\lambda \\ -12 + 6\lambda \end{pmatrix} \bullet \begin{pmatrix} -6 \\ 9 \\ 6 \end{pmatrix} = 0$$

$\Rightarrow -6(-2 - 6\lambda) + 9(-5 + 9\lambda) + 6(-12 + 6\lambda) = 0$

$\Rightarrow 12 + 36\lambda - 45 + 81\lambda - 72 + 36\lambda = 0$

$\Rightarrow -105 + 153\lambda = 0$

$$\Rightarrow \lambda = \frac{105}{153} = \frac{35}{51}$$

$$\overrightarrow{PP'} = \begin{pmatrix} -2-6\lambda \\ -5+9\lambda \\ -12+6\lambda \end{pmatrix} == \begin{pmatrix} -2-6\times\frac{35}{51} \\ -5+9\times\frac{35}{51} \\ -12+6\times\frac{35}{51} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-104}{17} \\ \frac{20}{17} \\ \frac{-134}{17} \end{pmatrix} == \frac{1}{17}\begin{pmatrix} -104 \\ 20 \\ -134 \end{pmatrix}$$

$$\Rightarrow PP' = \frac{1}{17}\sqrt{29172} = 10.04$$

The distance is 10.04 units

The intersection of three planes

To solve the intersection,
use the equations of the plane ax +by +cz +d = 0
to form an augmented matrix, which is solved
for x, y and z.

The intersection between three planes could be:

A single point

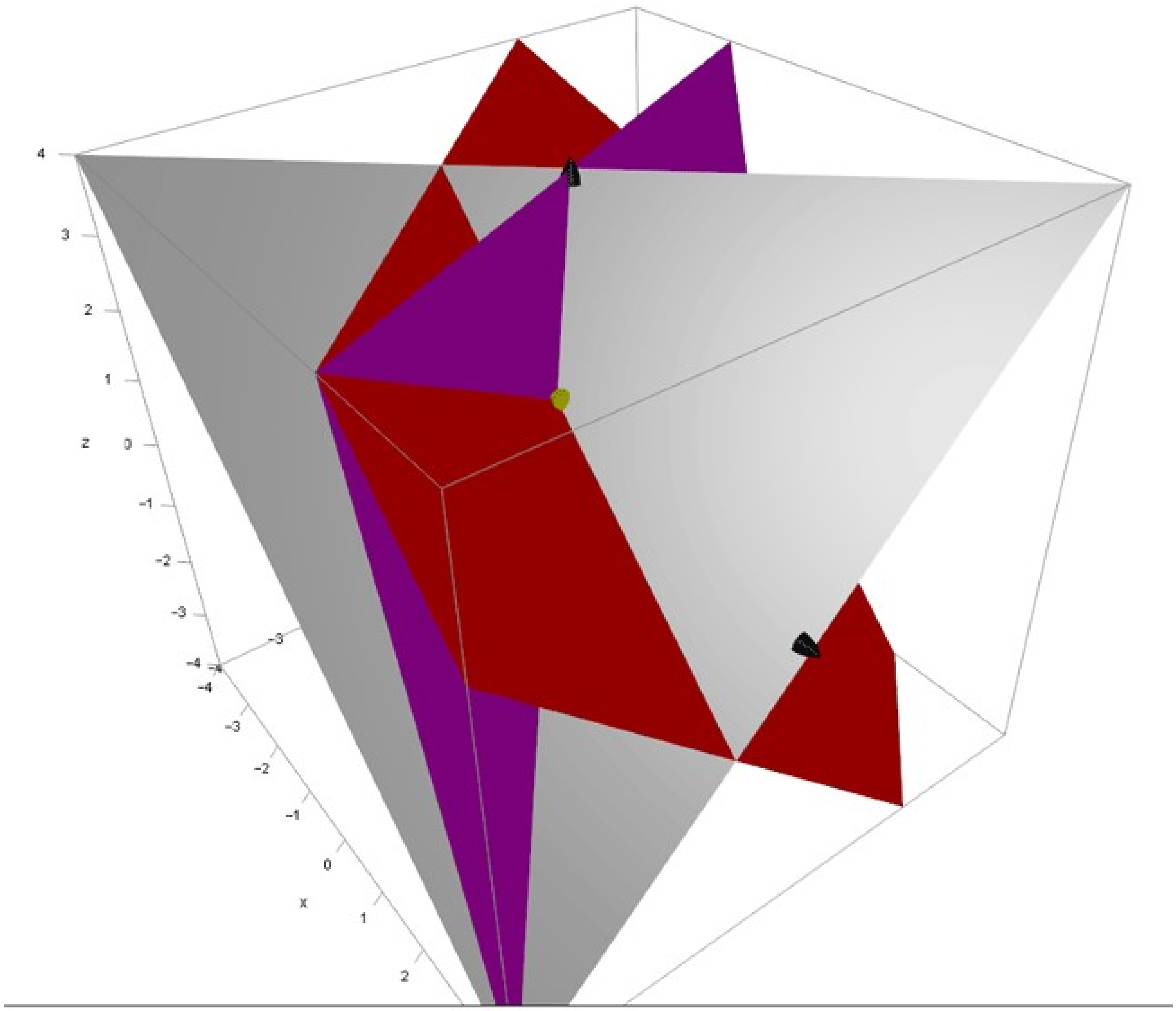
A unique solution is found

Example

$$x + y + z = 2$$
$$4x + 2y + z = 4$$
$$x - y + z = 4$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$Point \quad (1, -1, \; 2)$$



A line of intersection

An infinite number of solutions exist

Example

$$x + 2y + 2z = 11$$
$$x - y + 3z = 8$$
$$4x - y + 11z = 35$$

$$\begin{pmatrix} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

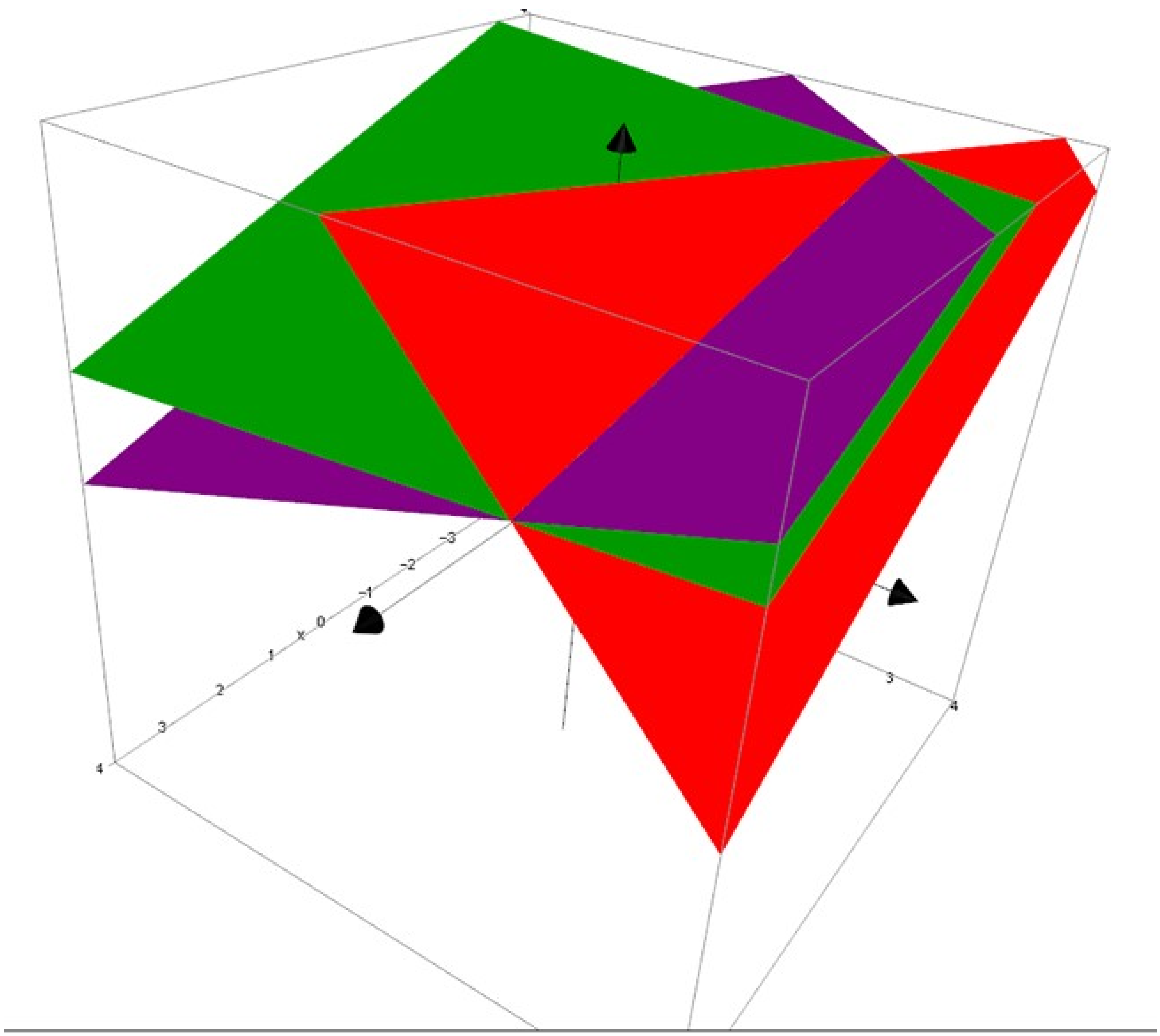
$$x + 2y + 2z = 11$$

$$x = 11 - 2y - 2z$$

$$x = 11 - 2\left(\frac{z+3}{3}\right) - 2z \quad -3y + z = -3$$

$$x = 13 - \frac{8z}{3} = \frac{39 - 8z}{3} \quad \Rightarrow y = \frac{z+3}{3} \quad z = z$$

Parametric equations



Two lines of intersection

An infinite number of solutions

Example

$$2x + 4y + 6z = 22$$
$$3y + 3z = -9$$
$$x + 2y + 3z = 16$$

which reduces to

$$\begin{pmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

The system is inconsistent

Using the second row

$$\text{let } z = t$$

$$so$$

$$y + t = -3$$

$$y = -3 - t$$

Substitute into first row

$$x + 2y + 3z = 11$$

$$x + 2(-3 - t) + 3t = 11$$

$$x - 6 - 2t + 3t = 11$$

$$x + t = 17$$

$$x = 17 - t$$

$$so$$

$$t = z = 17 - x = -y - 3$$

Substitute into third equation

$$x + 2y + 3z = 16$$

$$x + 2(-3 - t) + 3t = 16$$

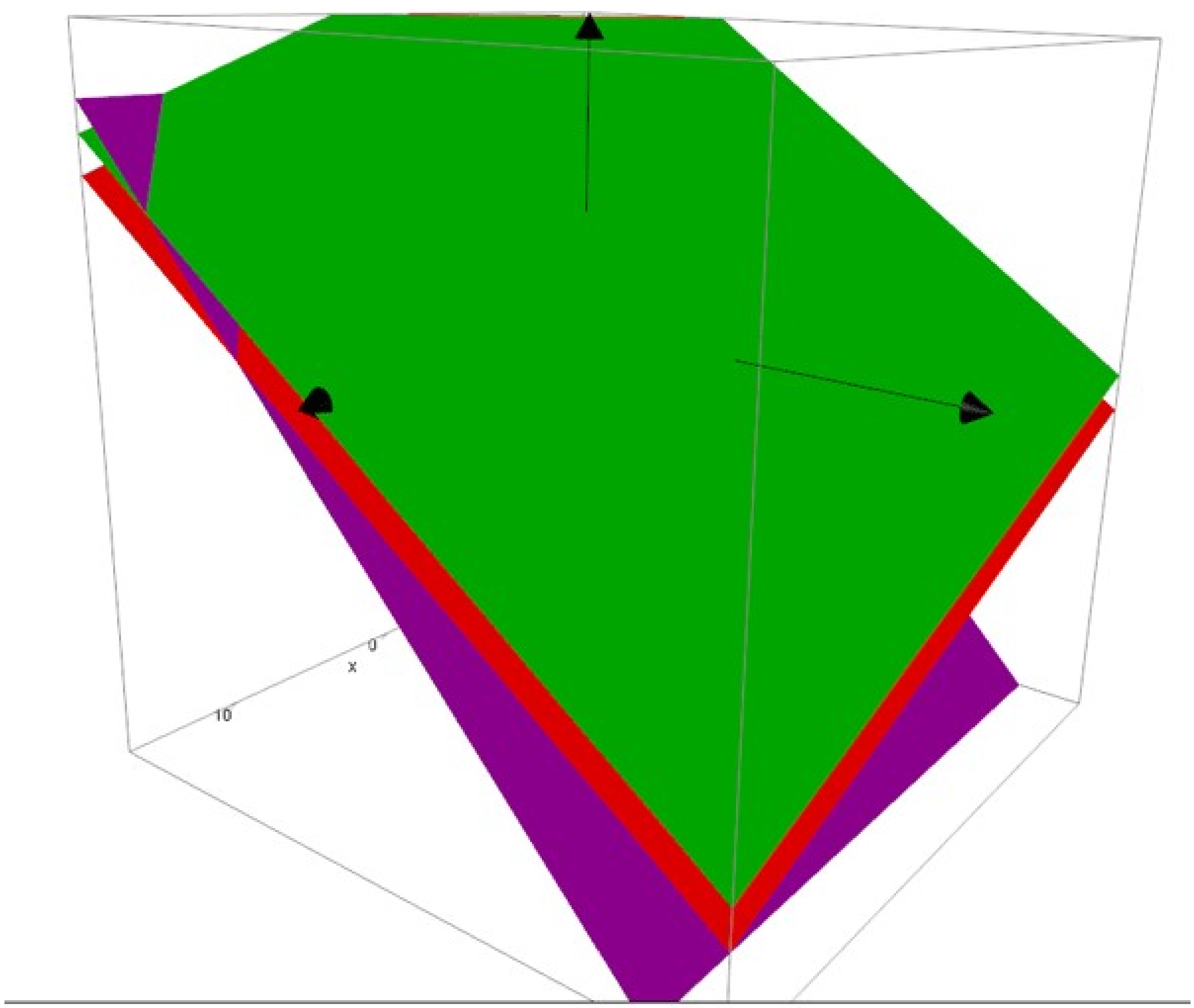
$$x - 6 - 2t + 3t = 16$$

$$x + t = 22$$

$$t = 22 - x$$

so

$$t = z = 22 - x = -y - 3$$



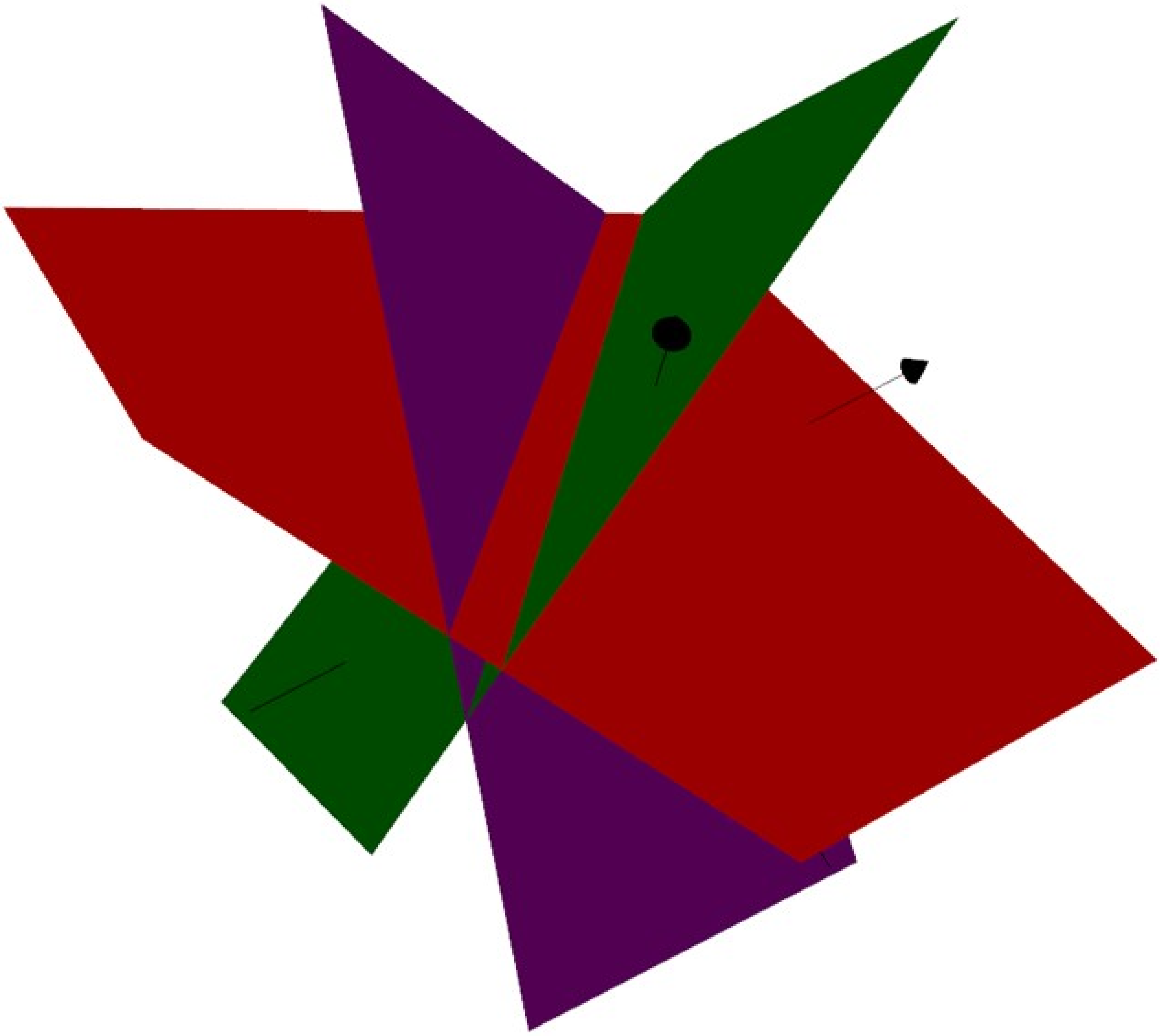
Three lines of intersection
Similar to above.
Examine each pair of planes in turn.

Example

$$3x - y + 2z = 1$$
$$x - 2y - z = -3$$
$$2x + y + 3z = 5$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



A plane of intersection

Two redundant equations

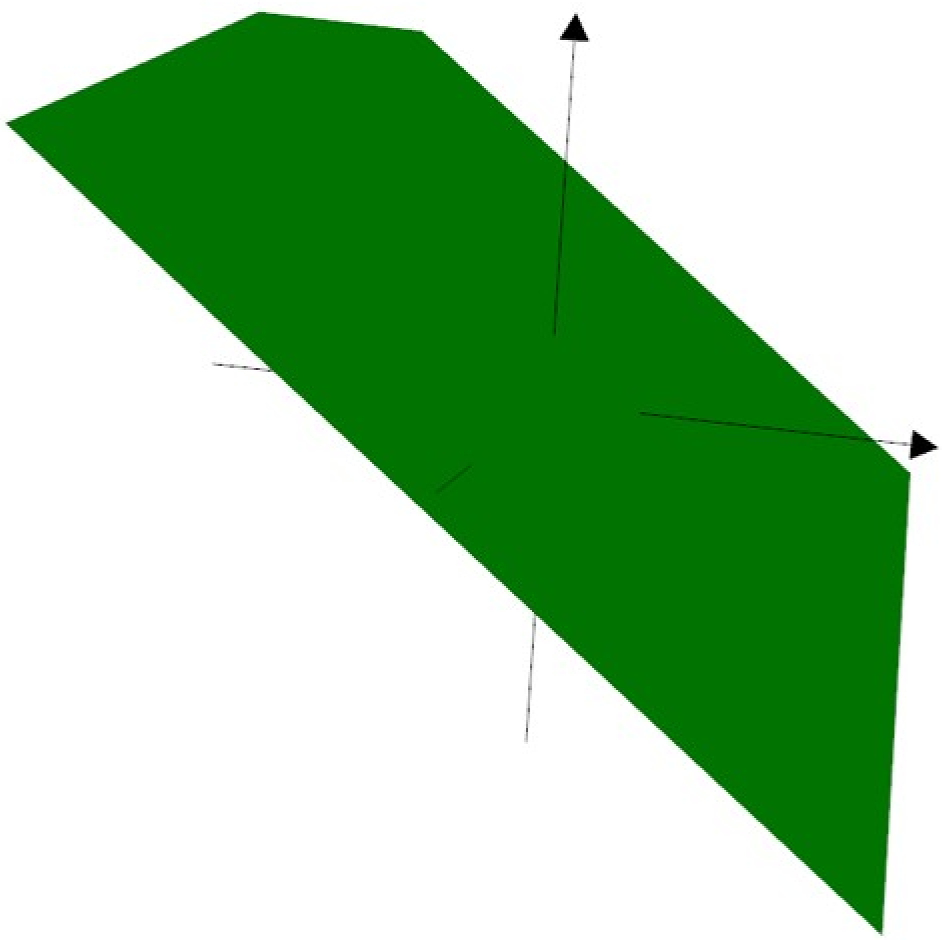
Example

$$3x - y + 4z = 3$$
$$6x - 2y + 8z = 6$$
$$15x - 5y + 20z = 15$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

No consistency



No intersection

Example

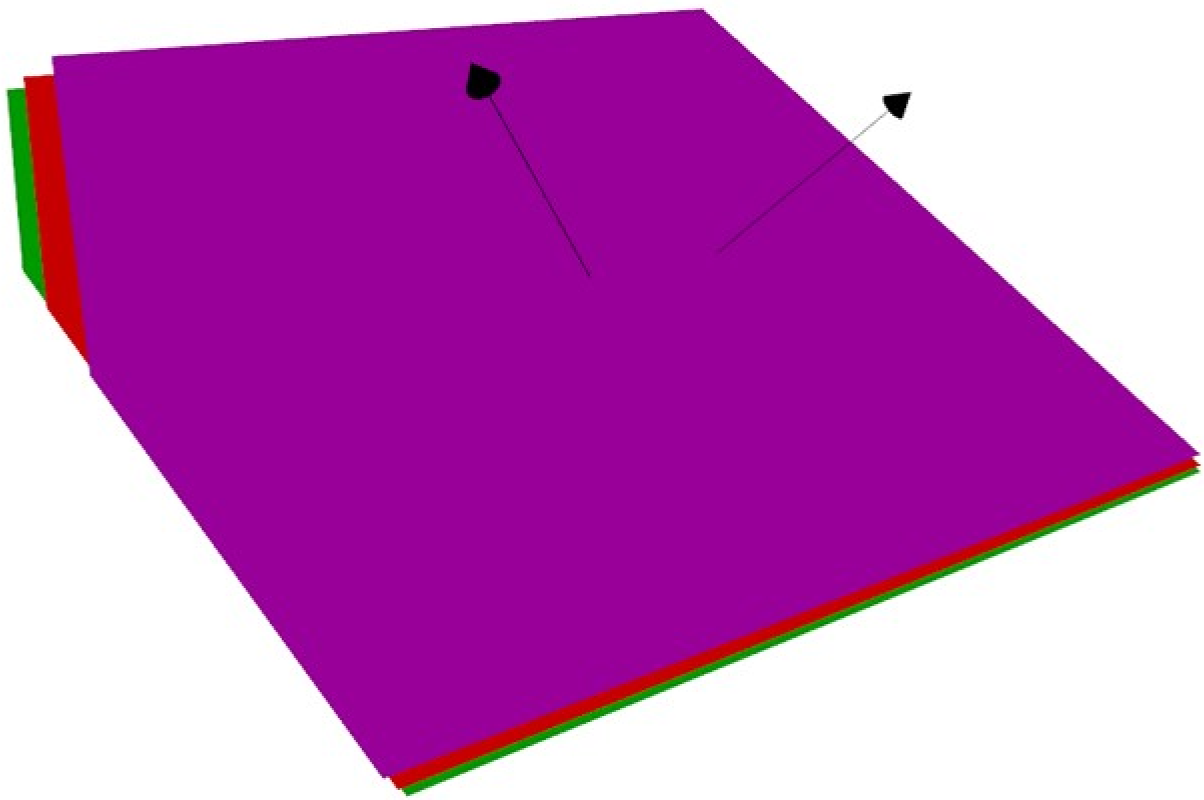
$$3x - y + 4z = 3$$
$$6x - 2y + 8z = 8$$
$$15x - 5y + 20z = 12$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

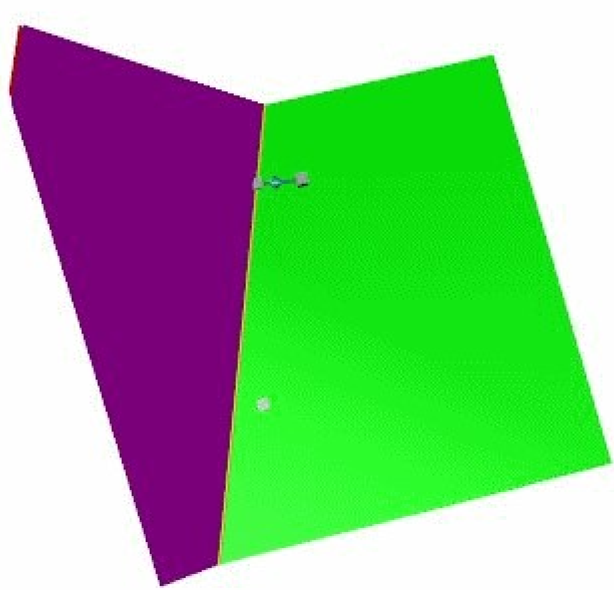
No consistency

All planes are parallel



Vector Equations

The angle between two planes



The angle between two planes is found
using the scalar product.
It is equal to the acute angle determined by
the normal vectors of the planes.

Example

Calculate the angle between the planes

π_1: $x + 2y - 2z = 5$

and π_2: $6x - 3y + 2z = 8$

$$\text{let } \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \text{ represent the normal for } \pi_1$$

$$\text{and } \quad \mathbf{b} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} \text{ represent the normal for } \pi_2$$

$$|\mathbf{a}| = \sqrt{1+4+4} \qquad\qquad |\mathbf{b}| = \sqrt{36+9+4}$$

$$= 3 \qquad\qquad\qquad = 7$$

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}||\mathbf{b}|}$$

$$\cos\theta = \frac{1\times 6 - 2\times 3 - 2\times 2}{21}$$

$$\cos\theta = \frac{-4}{21}$$

$$\theta = 100.98^\circ \quad \text{i.e obtuse}$$

$$\theta = 79.02^\circ$$

The distance between parallel planes

Let P be a point on plane π_1 : $ax + by + cz = n$

$\mathbf{a.x} = n$

and Q be a point on plane π_2 : ax + by + cz = m

a.x = m

Since the planes are parallel, they share the common normal, **a**
a=(a**i** +b**j**+c**k**)

The distance between the planes is

$$PQ = \frac{|m-n|}{|\mathbf{a}|}$$

Example

Calculate the distance between the planes

π_1: x + 2y - 2z = 5

and π_2: 6x + 12y - 12z = 8

$$x + 2y - 2z = 5$$

$$6x + 12y - 12z = 8$$

$$x + 2y - 2z = \frac{4}{3}$$

$$so \quad \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad n = 5 \quad \text{and} \quad \mathrm{m} = \frac{4}{3}$$

$$PQ = \frac{|m - n|}{|\mathbf{a}|}$$

$$= \frac{\left|\frac{4}{3} \quad - 5\right|}{\left|\sqrt{1 + 4 + 4}\right|}$$

$$= \frac{\frac{11}{3}}{3}$$

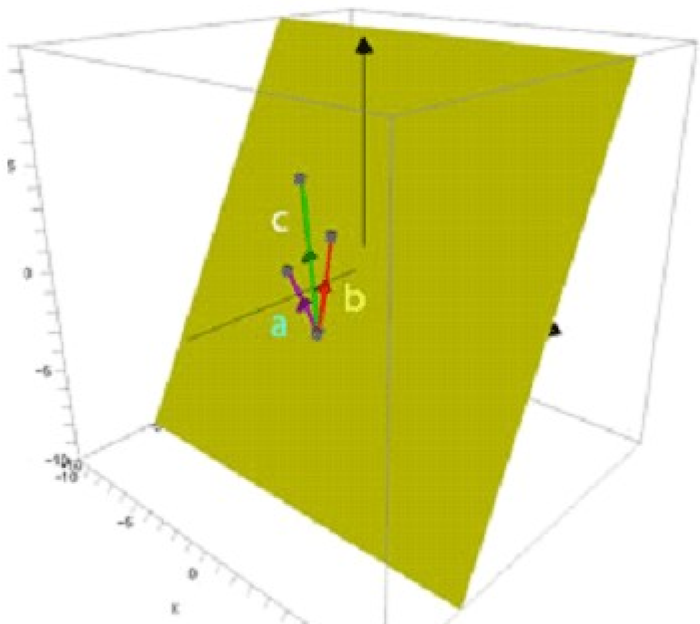
$$= \frac{11}{9}$$

$$= 1\frac{2}{9} \text{ units}$$

Coplanar vectors

If a relationship exists between the vectors **a**, **b** and **c** such that **c=**λ**a+**μ**b, where** λ and μ are constants**,**
then vectors a, b and c are co-planar.

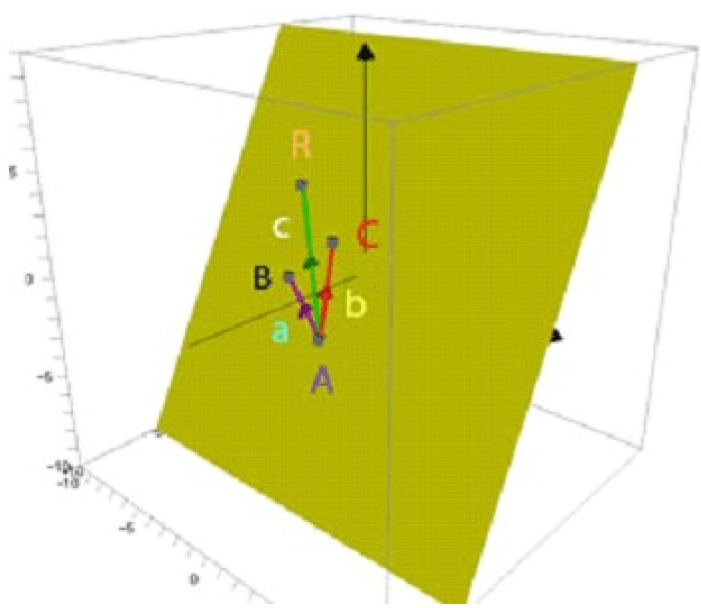
If three vectors are co-planar,
c=λ**a+**μ**b**



Vector equation of a plane

From the coplanar section above,
c=λa+μb

When position vectors are used,



$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$$

$$\overrightarrow{AR} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

$$\mathbf{r} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) + \mathbf{a}$$

$$\mathbf{r} = \lambda\mathbf{b} - \lambda\mathbf{a} + \mu\mathbf{c} - \mu\mathbf{a} + \mathbf{a}$$

$$\mathbf{r} = \mathbf{a} - \lambda\mathbf{a} - \mu\mathbf{a} + \mu\mathbf{c} + \lambda\mathbf{b}$$

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

r=(1-λ-u)**a**+ λ**b+**μ**c** is the **vector equation of the plane.**

Since λ and b are variable, there will be many possible equations for the plane.

Effects of changing λ and μ

Example

Find a vector equation of the plane through the points
A (-1,-2,-3) , B(-2,0,1) and C (-4,-1,-1)

$$\mathbf{r} = (1-\lambda-\mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

$$= (1-\lambda-\mu)\begin{pmatrix}-1\\-2\\-3\end{pmatrix} + \lambda\begin{pmatrix}-2\\0\\1\end{pmatrix} + \mu\begin{pmatrix}-4\\-1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-(1-\lambda-\mu)-2\lambda-4\mu\\-2(1-\lambda-\mu)-\mu\\-3(1-\lambda-\mu)+\lambda-\mu\end{pmatrix}$$

$$= \begin{pmatrix}-1+\lambda+\mu-2\lambda-4\mu\\-2+2\lambda+2\mu-\mu\\-3+3\lambda+3\mu+\lambda-\mu\end{pmatrix}$$

$$= \begin{pmatrix}-1-\lambda-3\mu\\-2+2\lambda+\mu\\-3+4\lambda+2\mu\end{pmatrix}$$

$$= (-1-\lambda-3\mu)\mathbf{i} + (-2+2\lambda+\mu)\mathbf{j} + (-3+4\lambda+2\mu)\mathbf{k}$$

If λ = 2 and μ =3

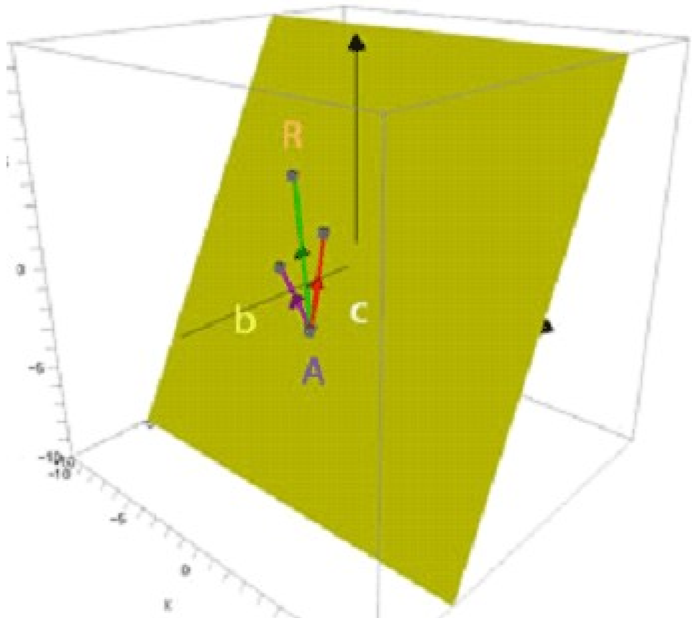
$$\mathbf{r} = (-1-\lambda-3\mu)\mathbf{i} + (-2+2\lambda+\mu)\mathbf{j} + (-3+4\lambda+2\mu)\mathbf{k}$$

$$\mathbf{r} = (-1-2-9)\mathbf{i} + (-2+4+3)\mathbf{j} + (-3+8+6)\mathbf{k}$$

$$\mathbf{r} = -12\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}$$

When A is a known point on the plane,
R is any old point on the plane and **b** and **c** are vectors parallel to the plane,

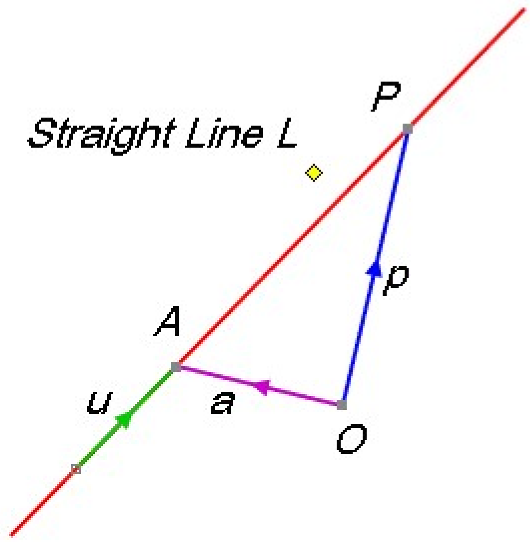
the **vector equation of the plane** is
r=a+λ**b+**μ**c**



The equations of a line

A line can be described when a point on it and its direction vector – a vector parallel to the line – are known.

In the diagram below, the line L passes through points $A(x_1,y_1,z_1)$ and P (x,y,z).



uis the direction vector a**i** +b**j** +c**k**
Being on the line, it has the same direction as any parallel line.

O is the origin.
a and **p** represent the position vectors of A and P.

P is on line L

$\Rightarrow \quad \overrightarrow{AP} = \lambda \mathbf{u}$ for some scalar λ

$\Rightarrow \quad \mathbf{p} - \mathbf{a} = \lambda \mathbf{u}$

$\Rightarrow \quad \mathbf{p} = \mathbf{a} + \lambda \mathbf{u}$

$$\mathbf{p} = \mathbf{a} + \lambda \mathbf{u}$$

is the vector equation of the line

convention often replaces **p** with **r**

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$

If two points are known, say A and B

then $\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} - \lambda\mathbf{a}$

$\Rightarrow \qquad \mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$

In component form, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 + \lambda a \\ y_1 + \lambda b \\ z_1 + \lambda c \end{pmatrix}$$

giving the parametric equations

$$x = x_1 + \lambda a \quad , \quad y = y_1 + \lambda b \quad , z = z_1 + \lambda c$$

so

$$\frac{x - x_1}{a} = \lambda \qquad \frac{y - y_1}{b} = \lambda \qquad \frac{z - z_1}{c} = \lambda$$

Giving the symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

This is also known as :

standard form,

canonical form,

co-ordinate equation

Example

Find the vector equation of the straight line through (3,2,1) which is parallel to the vector 2**i** +3**j** +4**k**

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$$

$$\Rightarrow \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

are the vector equations of the line

Example

Find the vector form of the equation of the straight line which has parametric equations

$$x = 4 - 2\lambda \qquad y = 7 + \lambda \qquad z = 3 - 4\lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{r} = 4\mathbf{i} + 7\mathbf{j} + 3\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$$

Example

Find the Cartesian form of the line which has position vector 3**i** +2**j** +**k** and is parallel to the vector **i** - **j** + **k**

$$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\Rightarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore x = 3 + \lambda \qquad y = 2 - \lambda \qquad z = 1 + \lambda$$

$$\frac{x-3}{1} = \frac{y-2}{-1} = \frac{z-1}{1} = \lambda$$

$$\Rightarrow \quad x - 3 = 2 - y = z - 1 = \lambda$$

Example

Find the vector equation of the line passing through A(1,2,3) and B(4,5,6)

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$$

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$\Rightarrow \quad \mathbf{u} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\Rightarrow \qquad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

alternatively

$$\mathbf{r} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$$

$$\Rightarrow \mathbf{r} = (1-\lambda)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\Rightarrow \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\Rightarrow \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

Example

The vector equation of a line is

$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

State the point with z co-ordinate 3 which also lies on this line.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + 2\lambda \qquad y = 2 - \lambda \qquad z = 6 + 3\lambda$$

When $z = 3$

$$3 = 6 + 3\lambda$$

$$\Rightarrow \quad \lambda = \frac{3-6}{3} = -1$$

$$\Rightarrow \quad x = 3 - 2 = 1 \qquad y = 2 + 1 = 3 \qquad z = 6 - 3 = 3$$

$\Rightarrow$ point $(1, 3, 3)$ lies on line

Example

A line L has equations

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{3-z}{4}$$

Is the vector $\mathbf{s} = 6\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$ parallel to L ?

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{3-z}{4} = \lambda$$

$$\Rightarrow \quad x = -2 + 3\lambda \qquad y = 1 + 2\lambda \qquad z = 3 - 4\lambda$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$$

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$$

(-2,1,3) is a point on L

and $\lambda(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ is a direction vector.

s has direction ratio $6 : 4 : -8 = 3 : 2 : -4$

The direction ratios of **s** and **u** are the same

$\Rightarrow \quad \mathbf{s} \parallel \mathbf{u}$

The angle between a line and a plane

The angle θ between a line and a plane is the complement of the angle between the line and the normal to the plane.

If the line has direction vector **u** and the normal to the plane is **a,** then

$$\sin\theta^\circ = \left|\cos(90-\theta)^\circ\right| = \frac{|\mathbf{a}\bullet\mathbf{u}|}{|\mathbf{a}||\mathbf{u}|}$$

Example

Given the equations

$$\frac{x-4}{3}=\frac{y-3}{2}=\frac{z-5}{6}$$

and the plane $6x+3y-2z=14$

1) Find the point of intersection

2) Find the angle the line makes with the plane.

1)

$$\frac{x-4}{3}=\frac{y-3}{2}=\frac{z-5}{6}=\lambda$$

$$\Rightarrow \quad x=4+3\lambda \qquad y=3+2\lambda \qquad z=5+6\lambda$$

$\therefore (4+3\lambda\ ,\ 3+2\lambda\ ,5+6\lambda)$ lies on the plane

$$6(4+3\lambda)+3(3+2\lambda)-2(5+6\lambda)=14$$

$$24+18\lambda+9+6\lambda-10-12\lambda=14$$

$$23+12\lambda=14$$

$$\lambda=\frac{14-23}{12}=\frac{-3}{4}$$

$$x=4+3\times\frac{-3}{4} \qquad y=3+2\times\frac{-3}{4} \qquad z=5+6\times\frac{-3}{4}$$

$$x=\frac{16-9}{4} \qquad y=\frac{12-6}{4} \qquad z=\frac{20-18}{4}$$

$$x=\frac{7}{4} \qquad y=\frac{3}{2} \qquad z=\frac{1}{2}$$

The point of intersection is $\left(\frac{7}{4}, \frac{3}{2}, \frac{1}{2}\right)$

2)

$$\sin\theta° = \left|\cos(90-\theta)°\right| = \frac{|\mathbf{a}\cdot\mathbf{u}|}{|\mathbf{a}||\mathbf{u}|}$$

$$\mathbf{a} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$
$$\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

$$\sin\theta° = \frac{\left|(6\mathbf{i}+3\mathbf{j}-2\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}+6\mathbf{k})\right|}{\left(\left|\sqrt{36+9+4}\right|\right)\left|\left(\sqrt{9+4+36}\right)\right|}$$

$$\Rightarrow \quad \sin\theta° = \frac{12}{49} \qquad (o \le \theta \le 90)$$
$$\Rightarrow \quad \theta = 14.175°$$

The angle of intersection is 14.2°

The intersection of two lines

Example

Show that the lines with equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\frac{x+1}{12} = \frac{y-7}{6} = \frac{z-5}{3} = \lambda_2$$

intersect and find the point of intersection and the equation of the plane containing the lines.

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + 4\lambda_1 \qquad y = 4 + \lambda_1 \qquad z = 1$$

and

$$\frac{x+1}{12} = \frac{y-7}{6} = \frac{z-5}{3} = \lambda_2$$

$$\Rightarrow \quad x = -1 + 12\lambda_2 \qquad y = 7 + 6\lambda_2 \qquad z = 5 + 3\lambda_2$$

Equating co-ordinates

$3+4\lambda_1 = -1+12\lambda_2 \qquad 4+\lambda_1 = 7+6\lambda_2 \qquad 1 = 5+3\lambda_2$

$4\lambda_1 = -4+12\lambda_2 \qquad (1)$

$\lambda_1 = 3+6\lambda_2 \qquad (2)$

$0 = 4+3\lambda_2 \qquad (3)$

From (3), $3\lambda_2 = -4$

$\Rightarrow \lambda_2 = \dfrac{-4}{3}$

$\Rightarrow \lambda_1 = 3+6\times\dfrac{-4}{3} = -5$

substituting

$$x = 3+4\lambda_1 \qquad y = 4+\lambda_1 \qquad z = 1$$

$$= 3-20 \qquad = 4-5$$

$$= -17 \qquad = -1$$

Intersection point is (-17, -1,1)

Let A(-17,-1,1) B(3,4,1) C(-1,7,5) be the points from the lines above

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ 4 \end{pmatrix}$$

$$\mathbf{n} \bullet \overrightarrow{\mathrm{AP}} = 0$$

Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{p} = \overrightarrow{OP}$, so $\overrightarrow{\mathrm{AP}} = \overrightarrow{OP} - \overrightarrow{OA}$

$$\Rightarrow \quad \mathbf{n} \bullet (\mathbf{p} - \mathbf{a}) = 0$$

Here, $\mathbf{n} = \left| \overrightarrow{AB} \times \overrightarrow{AC} \right|$

$$\mathbf{n} = \left| \overrightarrow{AB} \times \overrightarrow{AC} \right| = \left| \begin{pmatrix} i & j & k \\ 20 & 5 & 0 \\ 16 & 8 & 4 \end{pmatrix} \right|$$

$$= 20\mathbf{i} - 80\mathbf{j} + 80\mathbf{k}$$

$$= \mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{p} - \mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -17 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x+17 \\ y+1 \\ z-1 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{n} \bullet \overrightarrow{AP} = 0$$

$$\Rightarrow \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} x+17 \\ y+1 \\ z-1 \end{pmatrix} = 0$$

$$\Rightarrow x + 17 - 4(y+1) + 4(z-1) = 0$$

$$\Rightarrow x + 17 - 4y - 4 + 4z - 4 = 0$$

$$\Rightarrow x - 4y + 4z + 9 = 0$$

The intersection of two planes

To find the equations of the line of intersection of two planes, a direction vector and point on the line is required.

Since the line of intersection lies in both planes, the direction vector is parallel to the vector products of the normal of each plane.

Example

Find the equation for the line of intersection of the planes

-3x + 2y + z = -5
7x + 3y - 2z = -2

$$-3x+2y+z=-5 \qquad\qquad 7x+3y-2z=-2$$

Let $z=0$

Then $\quad -3x+2y=-5.....(1)$

and $\quad 7x+3y=-2.....(2)$

$(2)\times 2 \quad 14x+6y=-4$

$(1)\times -3 \quad 9x-6y=15$

add $\quad 23x=11$

$\Rightarrow \quad x=\dfrac{11}{23}$

subst in (1)

$$-\frac{33}{23}+2y=-5$$

$$\Rightarrow \quad y=\frac{-5+\frac{33}{23}}{2}=\frac{-41}{23}$$

The point $\left(\dfrac{11}{23}, \dfrac{-41}{23}, 0\right)$ is on the line of intersection

Normal vectors are $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

and $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \left| \begin{pmatrix} i & j & k \\ -3 & 2 & 1 \\ 7 & 3 & -2 \end{pmatrix} \right|$$

$$= -7\mathbf{i} + \mathbf{j} - 23\mathbf{k}$$

$$\mathbf{r} = \begin{pmatrix} \frac{11}{23} \\ \frac{-41}{23} \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -7 \\ 1 \\ -23 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{23} \\ \frac{-41}{23} \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} \frac{7}{23} \\ \frac{-1}{23} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad x = \frac{11}{23} + \frac{7}{23}\lambda_1 \qquad y = \frac{-41}{23} - \frac{1}{23}\lambda_1 \qquad z = \lambda_1$$

The distance from a point to a plane

To find the distance of a point P to a plane

1. Find the equation of the projection PP' by using the normal to the plane and the point P.
2. Find the co-ordinates of P' , the intersection with the plane.
3. Apply the distance formula to PP'

Alternatively

The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example

Find the distance between the point (3,1,-2)
and the plane x +2y + 2z = - 4

$$\mathbf{r} = u + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad x = 3 + \lambda_1 \qquad y = 1 + 2\lambda_1 \qquad z = -2 + 2\lambda_1$$

Plane equation is $x + 2y + 2z + 4 = 0$

$$\Rightarrow 3 + \lambda_1 + 2(1 + 2\lambda_1) + 2(-2 + 2\lambda_1) + 4 = 0$$

$$\Rightarrow 3 + \lambda_1 + 2 + 4\lambda_1 - 4 + 4\lambda_1 + 4 = 0$$

$$\Rightarrow 5 + 9\lambda_1 = 0$$

$$\Rightarrow \lambda_1 = \frac{-5}{9}$$

$$\Rightarrow \quad x = 3 - \frac{5}{9} \qquad y = 1 - \frac{10}{9} \qquad z = -2 - \frac{10}{9}$$

$$P'\left(\frac{22}{9}, -\frac{1}{9}, -\frac{28}{9}\right)$$

$$PP' = \begin{pmatrix} \frac{-5}{9} \\ -\frac{10}{9} \\ -\frac{10}{9} \end{pmatrix} = \frac{-5}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow |PP'| = \left|\frac{-5}{9}\sqrt{1 + 4 + 4}\right|$$

$$= \left|\frac{-5}{3}\right|$$

$$= \frac{5}{3} \text{ units}$$

Alternatively

$$x + 2y + 2z = -4 \qquad \text{at } (3,1,-2)$$

$$\Rightarrow \; x + 2y + 2z + 4 = 0$$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|3 + 2 - 4 + 4|}{\sqrt{1 + 4 + 4}}$$

$$= \frac{5}{3}$$

The distance is $\frac{5}{3}$ units

The distance from a point to a line

To find the distance of a point P to a Line L

1. Let the line have direction vector **u** and parameter λ
2. Find the co-ordinates of PP′ by using the scalar product with **u** and the point P.
3. Apply the distance formula to PP′

Find the distance between the line

$$\frac{x+3}{-6} = \frac{y-2}{9} = \frac{z+8}{6}$$

and the point P $(-1, 7, 4)$

$$P' = \begin{pmatrix} -3 \\ 2 \\ -8 \end{pmatrix} + \lambda_1 \begin{pmatrix} -6 \\ 9 \\ 6 \end{pmatrix}$$

$\Rightarrow \quad x = -3 - 6\lambda_1 \qquad y = 2 + 9\lambda_1 \qquad z = -8 + 6\lambda_1$

$P'(-3 - 6\lambda_1 \text{ , } 2+9\lambda_1 \text{ ,}-8+6\lambda_1)$

$$\overrightarrow{PP'} = \begin{pmatrix} -3 - 6\lambda \\ 2 + 9\lambda_1 \\ -8 + 6\lambda_1 \end{pmatrix} - \begin{pmatrix} -1 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 - 6\lambda \\ -5 + 9\lambda \\ -12 + 6\lambda \end{pmatrix}$$

$\overrightarrow{PP'} \bullet \mathbf{u} = 0$

$$\Rightarrow \quad \begin{pmatrix} -2 - 6\lambda \\ -5 + 9\lambda \\ -12 + 6\lambda \end{pmatrix} \bullet \begin{pmatrix} -6 \\ 9 \\ 6 \end{pmatrix} = 0$$

$\Rightarrow -6(-2 - 6\lambda) + 9(-5 + 9\lambda) + 6(-12 + 6\lambda) = 0$

$\Rightarrow 12 + 36\lambda - 45 + 81\lambda - 72 + 36\lambda = 0$

$\Rightarrow -105 + 153\lambda = 0$

$$\Rightarrow \lambda = \frac{105}{153} = \frac{35}{51}$$

$$\overrightarrow{PP'} = \begin{pmatrix} -2-6\lambda \\ -5+9\lambda \\ -12+6\lambda \end{pmatrix} == \begin{pmatrix} -2-6\times\frac{35}{51} \\ -5+9\times\frac{35}{51} \\ -12+6\times\frac{35}{51} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-104}{17} \\ \frac{20}{17} \\ \frac{-134}{17} \end{pmatrix} == \frac{1}{17}\begin{pmatrix} -104 \\ 20 \\ -134 \end{pmatrix}$$

$$\Rightarrow PP' = \frac{1}{17}\sqrt{29172} = 10.04$$

The distance is 10.04 units

The intersection of three planes

To solve the intersection,
use the equations of the plane ax +by +cz +d = 0
to form an augmented matrix, which is solved
for x, y and z.

The intersection between three planes could be:

A single point

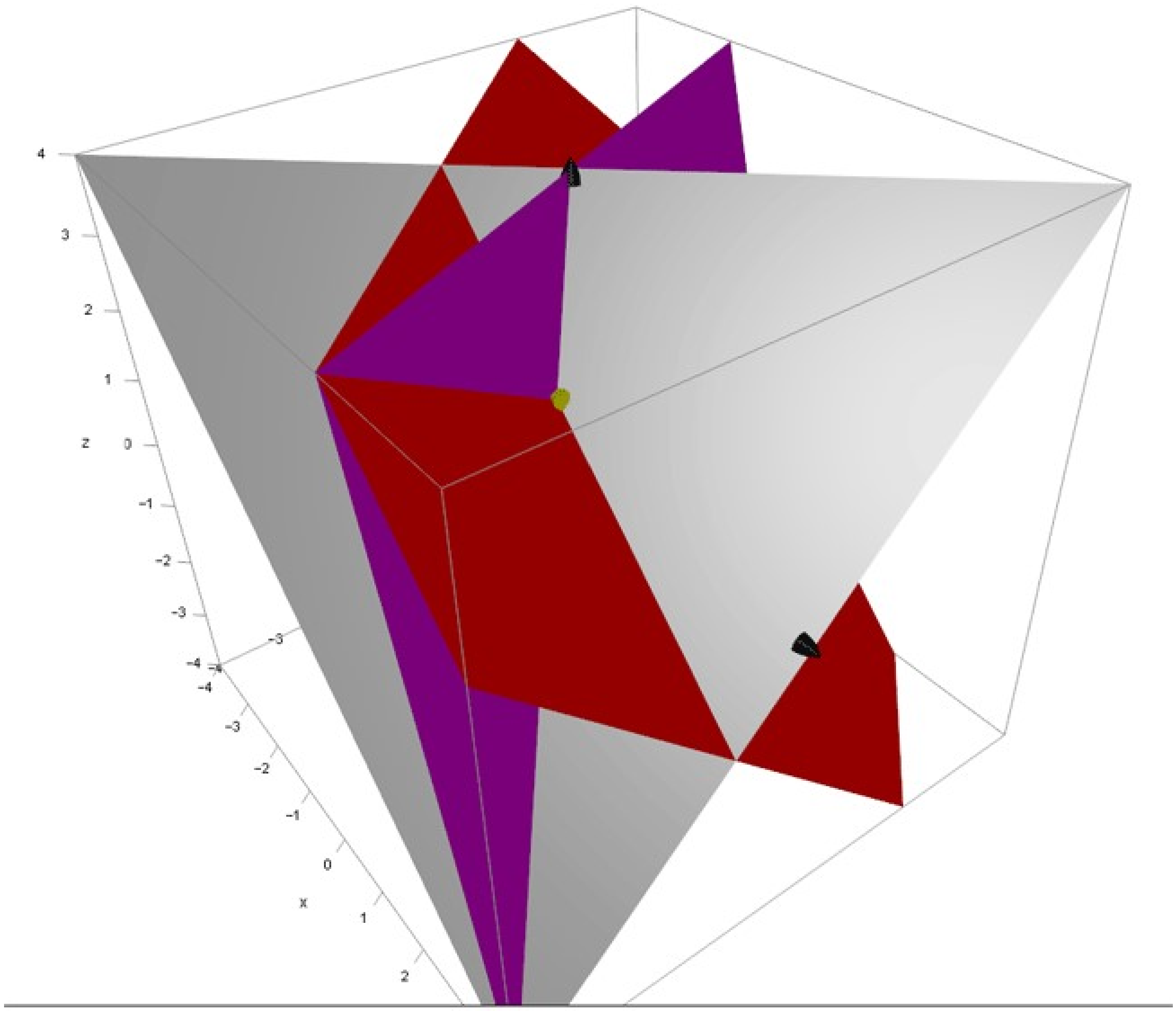
A unique solution is found

Example

$$x + y + z = 2$$
$$4x + 2y + z = 4$$
$$x - y + z = 4$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$Point \quad (1, -1, \; 2)$$



A line of intersection

An infinite number of solutions exist

Example

$$x + 2y + 2z = 11$$
$$x - y + 3z = 8$$
$$4x - y + 11z = 35$$

$$\begin{pmatrix} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

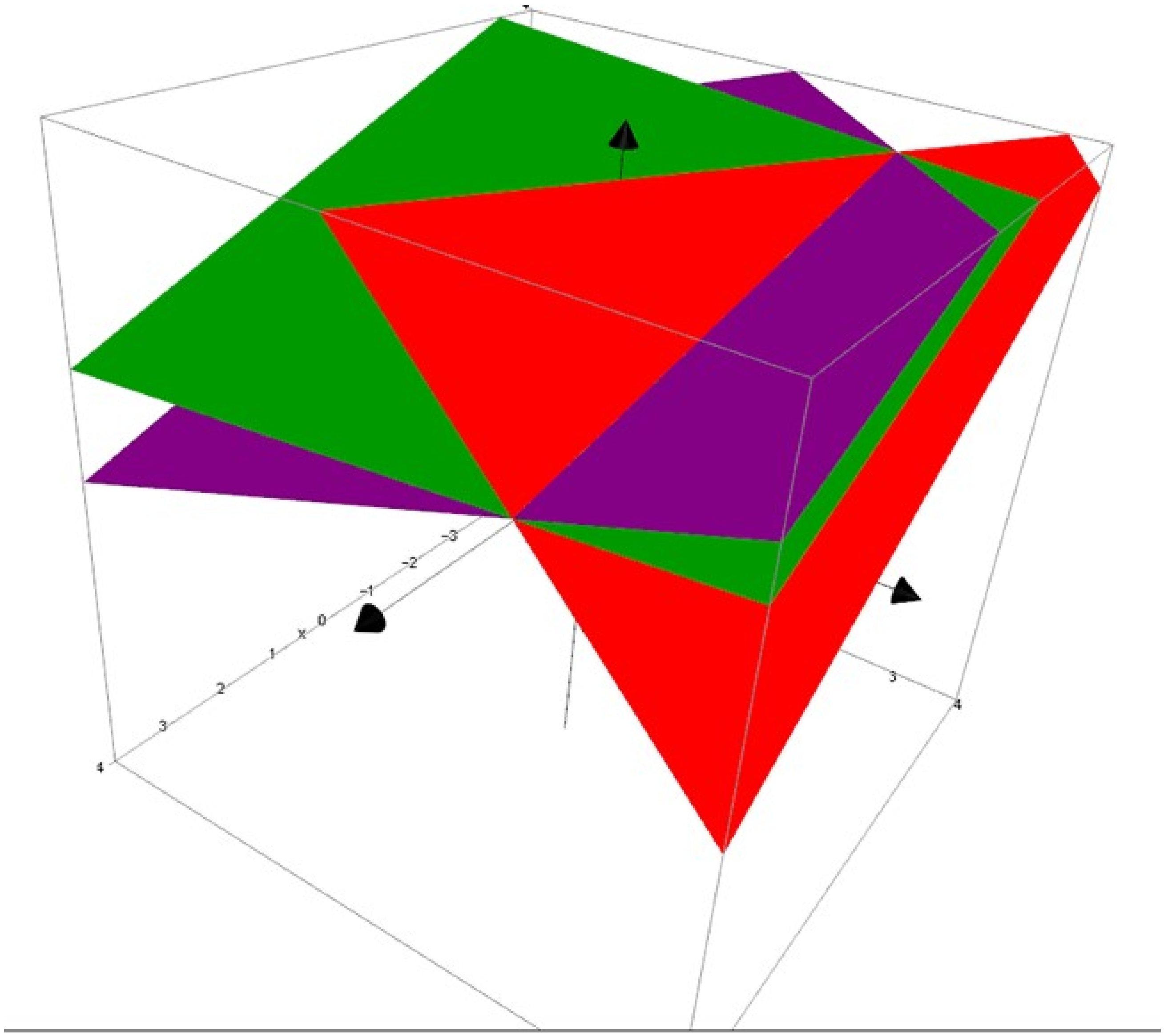
$$x + 2y + 2z = 11$$

$$x = 11 - 2y - 2z$$

$$x = 11 - 2\left(\frac{z+3}{3}\right) - 2z \qquad -3y + z = -3$$

$$x = 13 - \frac{8z}{3} = \frac{39 - 8z}{3} \qquad \Rightarrow y = \frac{z+3}{3} \qquad z = z$$

Parametric equations



Two lines of intersection

An infinite number of solutions

Example

$$2x + 4y + 6z = 22$$
$$3y + 3z = -9$$
$$x + 2y + 3z = 16$$

which reduces to

$$\begin{pmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

The system is inconsistent

Using the second row

$$\text{let } z = t$$

$$so$$

$$y + t = -3$$

$$y = -3 - t$$

Substitute into first row

$$x + 2y + 3z = 11$$

$$x + 2(-3 - t) + 3t = 11$$

$$x - 6 - 2t + 3t = 11$$

$$x + t = 17$$

$$x = 17 - t$$

$$so$$

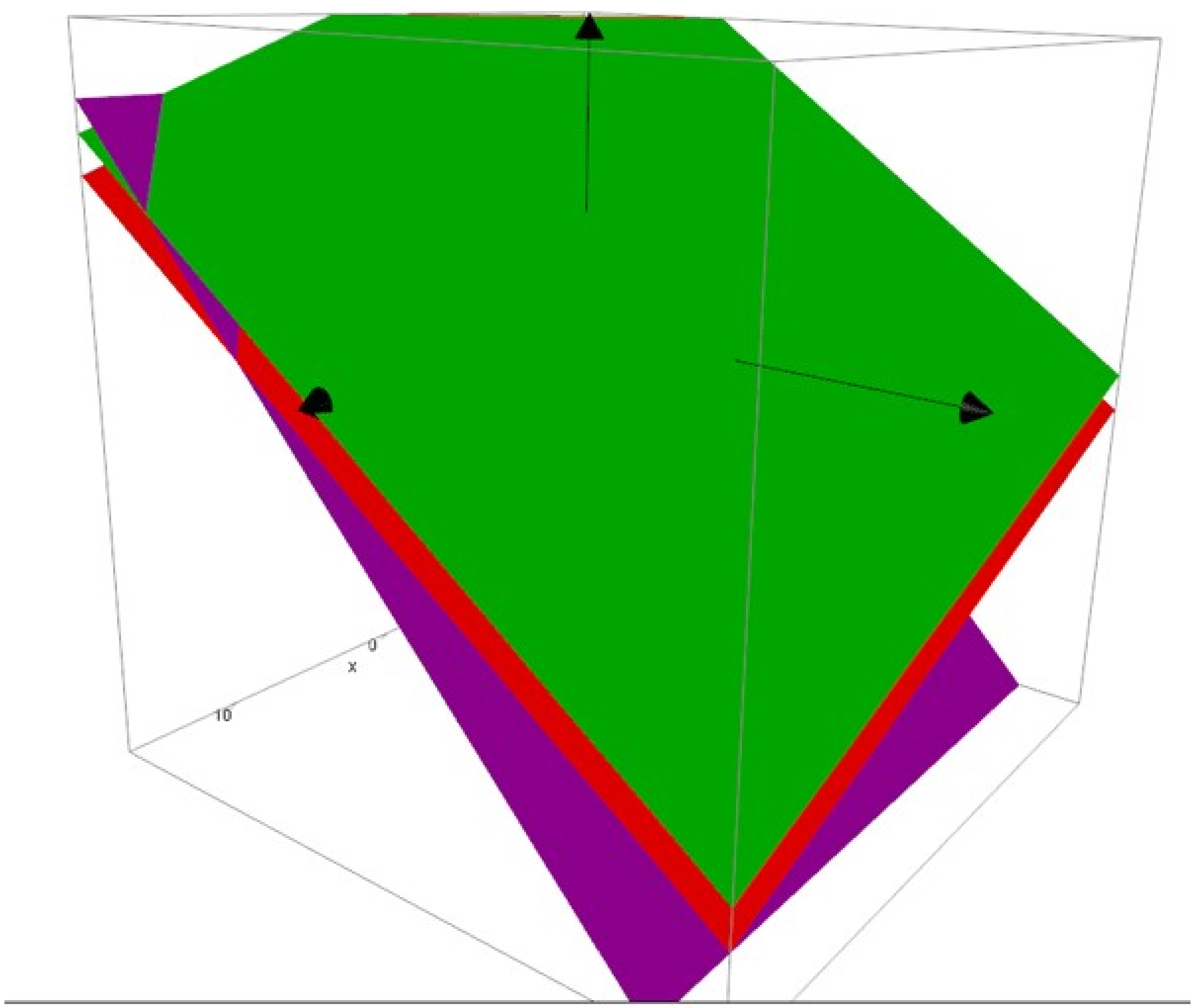
$$t = z = 17 - x = -y - 3$$

Substitute into third equation

$$x + 2y + 3z = 16$$
$$x + 2(-3 - t) + 3t = 16$$
$$x - 6 - 2t + 3t = 16$$
$$x + t = 22$$
$$t = 22 - x$$

so

$$t = z = 22 - x = -y - 3$$



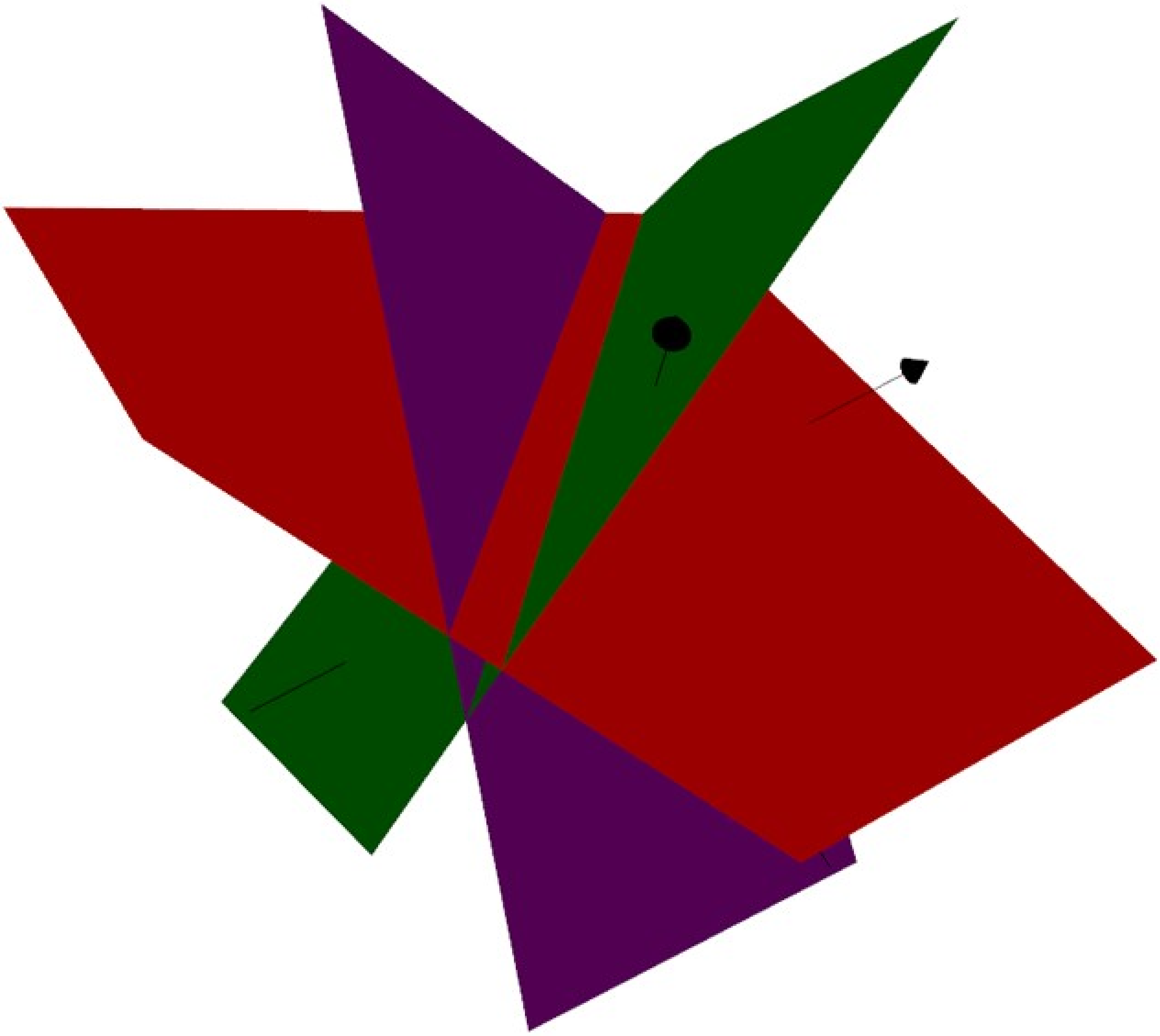
Three lines of intersection
Similar to above.
Examine each pair of planes in turn.

Example

$$3x - y + 2z = 1$$
$$x - 2y - z = -3$$
$$2x + y + 3z = 5$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



A plane of intersection

Two redundant equations

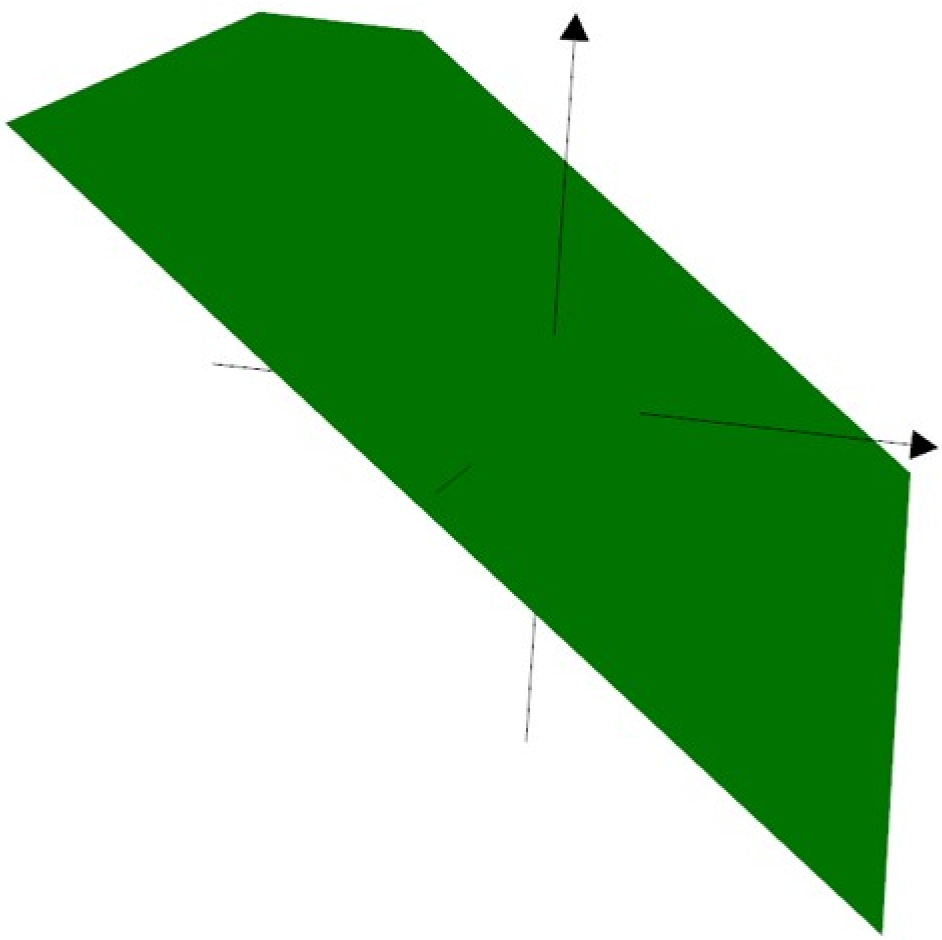
Example

$$3x - y + 4z = 3$$
$$6x - 2y + 8z = 6$$
$$15x - 5y + 20z = 15$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

No consistency



No intersection

Example

$$3x - y + 4z = 3$$
$$6x - 2y + 8z = 8$$
$$15x - 5y + 20z = 12$$

Which reduces to

$$\begin{pmatrix} 1 & \frac{-1}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

No consistency

All planes are parallel