Linear Combination of Vectors.

A vector $\mathbf{r}$ is said to be a linear combination of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$..... etc, if there exist scalars *x*, *y*, *z* etc., such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + ...$

Examples: Vectors $\mathbf{r}_1 = 2\mathbf{a} + \mathbf{b} + 3\mathbf{c}, \mathbf{r}_2 = \mathbf{a} + 3\mathbf{b} + \sqrt{2}\mathbf{c}$ are linear combinations of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(1) **Collinear and Non-collinear vectors:** Let $\mathbf{a}$ and $\mathbf{b}$ be two collinear vectors and let $\mathbf{x}$ be the unit vector in the direction of $\mathbf{a}$. Then the unit vector in the direction of $\mathbf{b}$ is $\mathbf{x}$ or $-\mathbf{x}$ according as $\mathbf{a}$ and $\mathbf{b}$ are like or unlike parallel vectors. Now, $\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{x}}$ and $\mathbf{b} = \pm|\mathbf{b}|\,\hat{\mathbf{x}}$.

$$\therefore \mathbf{a} = \left(\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)|\mathbf{b}|\,\hat{\mathbf{x}} \Rightarrow \mathbf{a} = \left(\pm\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)\mathbf{b} \Rightarrow \mathbf{a} = \lambda\mathbf{b}, \text{ where } \lambda = \pm\frac{|\mathbf{a}|}{|\mathbf{b}|}.$$

Thus, if $\mathbf{a}, \mathbf{b}$ are collinear vectors, then $\mathbf{a} = \lambda\,\mathbf{b}$ or $\mathbf{b} = \lambda\,\mathbf{a}$ for some scalar λ.

(2) **Relation between two parallel vectors**

(i) If $\mathbf{a}$ and $\mathbf{b}$ be two parallel vectors, then there exists a scalar *k* such that $\mathbf{a} = k\,\mathbf{b}$.

i.e., there exist two non-zero scalar quantities *x* and *y* so that $x\,\mathbf{a} + y\,\mathbf{b} = \mathbf{0}$.

If $\mathbf{a}$ and $\mathbf{b}$ be two non-zero, non-parallel vectors then $x\mathbf{a} + y\mathbf{b} = \mathbf{0} \Rightarrow x = 0$ and $y = 0$.

$$\text{Obviously } x\mathbf{a} + y\mathbf{b} = \mathbf{0} \Rightarrow \begin{cases} \mathbf{a} = \mathbf{0}, \mathbf{b} = \mathbf{0} \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \mathbf{a} \parallel \mathbf{b} \end{cases}$$

(ii) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then from the property of parallel vectors, we have

$$\mathbf{a} \parallel \mathbf{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

(3) **Test of collinearity of three points:** Three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear iff there exist scalars *x*, *y*, *z* not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$, where $x + y + z = 0$. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$, then the points with position vector $\mathbf{a}, \mathbf{b}, \mathbf{c}$ will be collinear iff $\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0$.

(4) **Test of coplanarity of three vectors:** Let $\mathbf{a}$ and $\mathbf{b}$ two given non-zero non-collinear vectors. Then any vectors $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be uniquely expressed as $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$ for some scalars *x* and *y*.

(5) **Test of coplanarity of Four points:** Four points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are coplanar iff there exist scalars *x*, *y*, *z*, *u* not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + u\mathbf{d} = \mathbf{0}$, where $x + y + z + u = 0$.

Four points with position vectors

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$

will be coplanar, *iff* $\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0$