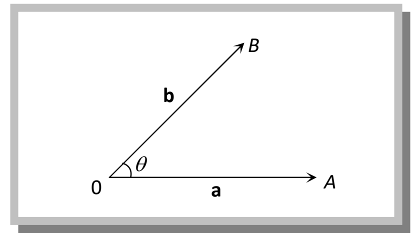
# Product of Two Vectors.

Product of two vectors is processed by two methods. When the product of two vectors results is a scalar quantity, then it is called scalar product. It is also known as dot product because we are putting a dot (.) between two vectors.

When the product of two vectors results is a vector quantity then this product is called vector product. It is also known as cross product because we are putting a cross (×) between two vectors.




(1) **Scalar or Dot product of two vectors :** If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors and $\theta$ be the angle between them, then their scalar product (or dot product) is denoted by $\mathbf{a}\,.\,\mathbf{b}$ and is defined as the scalar $|\,\mathbf{a}\,||\,\mathbf{b}\,|\,\cos\theta$, where $|\,\mathbf{a}\,|$ and $|\,\mathbf{b}\,|$ are modulii of $\mathbf{a}$ and $\mathbf{b}$ respectively and $0 \leq \theta \leq \pi$.

### Important Tips

☞ $\mathbf{a}\,.\,\mathbf{b} \in R$

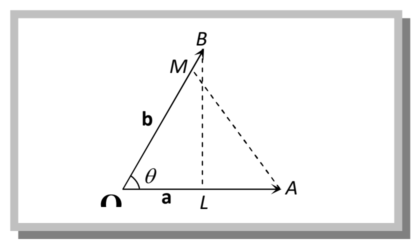
☞ $\mathbf{a}\,.\,\mathbf{b} \leq |\,\mathbf{a}\,||\,\mathbf{b}\,|$

☞ $\mathbf{a}\,.\,\mathbf{b} > 0 \Rightarrow$ *angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is acute.*

☞ $\mathbf{a}\,.\,\mathbf{b} < 0 \Rightarrow$ *angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is obtuse.*

☞ *The dot product of a zero and non-zero vector is a scalar zero.*

(i) **Geometrical Interpretation of scalar product**: Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let $\theta$ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw $BL \perp OA$ and $AM \perp OB$.




From $\Delta s$ $OBL$ and $OAM$, we have $OL = OB\cos\theta$ and $OM = OA\cos\theta$. Here $OL$ and $OM$ are known as projection of $\mathbf{b}$ on $\mathbf{a}$ and $\mathbf{a}$ on $\mathbf{b}$ respectively.

Now $\mathbf{a}\,.\,\mathbf{b} = |\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta = |\,\mathbf{a}\,|(OB\cos\theta) = |\,\mathbf{a}\,|(OL)$

$= (\text{Magnitude of } \mathbf{a})(\text{Projection of } \mathbf{b} \text{ on } \mathbf{a})$ .....(i)

Again, $\mathbf{a}\,.\,\mathbf{b} = |\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\theta = |\,\mathbf{b}\,|(|\,\mathbf{a}\,|\cos\theta) = |\,\mathbf{b}\,|(OA\cos\theta) = |\,\mathbf{b}\,|(OM)$

**a.b** = (Magnitude of **b**) (Projection of $\mathbf{a}$ on $\mathbf{b}$ ) .....(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

(ii) **Angle between two vectors :**If $\mathbf{a}, \mathbf{b}$ be two vectors inclined at an angle $\theta$, then,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

$$\Rightarrow \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; $\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$

(2) **Properties of scalar product**

(i) **Commutativity**:The scalar product of two vector is commutative *i.e.,* $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ .

(ii) **Distributivity of scalar product over vector addition:** The scalar product of vectors is distributive over vector addition *i.e.,*

(a) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (Left distributivity)

(b) $(\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}$ (Right distributivity)

(iii) Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$.

As $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are mutually perpendicular unit vectors along the co-ordinate axes, therefore

$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$; $\mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0$; $\mathbf{k} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = 0$.

(iv) For any vector $\mathbf{a}, \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

As $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the co-ordinate axes, therefore $\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}|^2 = 1$, $\mathbf{j} \cdot \mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}|^2 = 1$

(v) If *m* is a scalar and $\mathbf{a}, \mathbf{b}$ be any two vectors, then $(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$

(vi) If *m, n* are scalars and $\mathbf{a}, \mathbf{b}$ be two vectors, then $m\mathbf{a} \cdot n\mathbf{b} = mn(\mathbf{a} \cdot \mathbf{b}) = (mn\,\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (mn\,\mathbf{b})$

(vii) For any vectors $\mathbf{a}$ and $\mathbf{b}$, we have (a) $\mathbf{a} \cdot (-\mathbf{b}) = -(\mathbf{a} \cdot \mathbf{b}) = (-\mathbf{a}) \cdot \mathbf{b}$ (b) $(-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$

(viii) For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(a) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

(b) $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b}$

(c) $(\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$

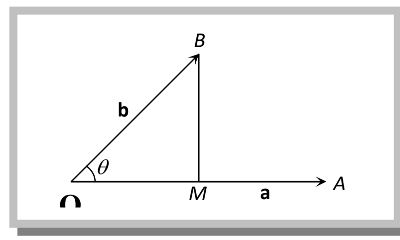
(d) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| \Rightarrow \mathbf{a} \parallel \mathbf{b}$

(e) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \Rightarrow \mathbf{a} \perp \mathbf{b}$

(f) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Rightarrow \mathbf{a} \perp \mathbf{b}$

(3) **Scalar product in terms of components**.If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, Then, $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. Thus, scalar product of two vectors is equal to the sum of the products of their corresponding components. In particular, $\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$.

(4) **Components of a vector along and perpendicular to another vector:** If $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Draw $BM \perp OA$. In $\Delta OBM$, we have $\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB} \Rightarrow \mathbf{b} = \overrightarrow{OM} + \overrightarrow{MB}$




Thus, $\overrightarrow{OM}$ and $\overrightarrow{MB}$ are components of $\mathbf{b}$ along $\mathbf{a}$ and perpendicular to $\mathbf{a}$ respectively.

Now, $\overrightarrow{OM} = (OM)\,\hat{\mathbf{a}} = (OB\cos\theta)\,\hat{\mathbf{a}} = (|\mathbf{b}|\cos\theta)\hat{\mathbf{a}} = \left(|\mathbf{b}|\dfrac{(\mathbf{a}.\mathbf{b})}{|\mathbf{a}||\mathbf{b}|}\right)\hat{\mathbf{a}} =$

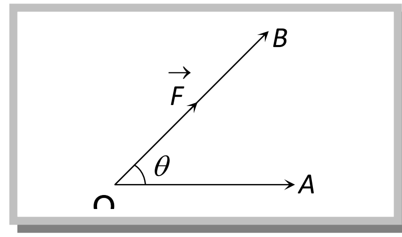
$$\left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|}\right)\hat{\mathbf{a}} = \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$

$$\therefore \mathbf{b} = \overrightarrow{OM} + \overrightarrow{MB} \Rightarrow \overrightarrow{MB} = \mathbf{b} - \overrightarrow{OM} = \mathbf{b} - \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$

Thus, the components of $\mathbf{b}$ along and perpendicular to $\mathbf{a}$ are $\left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ and $\mathbf{b} - \left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

respectively.

(5) **Work done by a force:** A force acting on a particle is said to do work if the particle is displaced in a direction which is not perpendicular to the force.




The work done by a force is a scalar quantity and its measure is equal to the product of the magnitude of the force and the resolved part of the displacement in the direction of the force.

If a particle be placed at $O$ and a force $\vec{F}$ represented by $\overrightarrow{OB}$ be acting on the particle at $O$. Due to the application of force $\vec{F}$ the particle is displaced in the direction of $\overrightarrow{OA}$ . Let $\overrightarrow{OA}$ be the displacement. Then the component of $\overrightarrow{OA}$ in the direction of the force $\vec{F}$ is $|\overrightarrow{OA}| \cos\theta$ .

$\therefore$ Work done = $|\vec{F}||\overrightarrow{OA}|\cos\theta = \vec{F}.\overrightarrow{OA} = \vec{F}.\mathbf{d}$ , where $\mathbf{d} = \overrightarrow{OA}$ Or Work done = (Force) . (Displacement)

If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force.