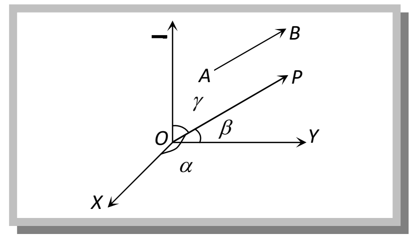
Direction cosines and Direction ratio.

(1) **Direction cosines**

(i) The cosines of the angle made by a line in anticlockwise direction with positive direction of co-ordinate axes are called the direction cosines of that line.



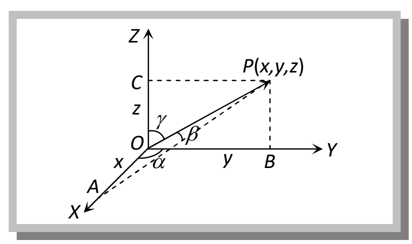
If α, β, γ be the angles which a given directed line makes with the positive direction of the x, y, z co-ordinate axes respectively, then cosα, cosβ, cosγ are called the direction cosines of the given line and are generally denoted by l, m, n respectively. Thus, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

By definition, it follows that the direction cosine of the axis of x are respectively $\cos 0^o$, $\cos 90^o$, $\cos 90^o$ i.e. (1, 0, 0). Similarly direction cosines of the axes of y and z are respectively (0, 1, 0) and (0, 0, 1).

Relation between the direction cosines: Let OP be any line through the origin O which has direction cosines l, m, n. Let P = (x, y, z) and OP = r. Then $OP^2 = x^2 + y^2 + z^2 = r^2$ .....(i)

From P draw PA, PB, PC perpendicular on the co-ordinate axes, so that

OA = x, OB = y, OC = z. Also, $\angle POA = \alpha, \angle POB = \beta$ and $\angle POC = \gamma$.



From triangle AOP, $l = \cos\alpha = \frac{x}{r} \Rightarrow x = lr$

Similarly $y = mr$ and $z = nr$.

Hence from (i), $r^2(l^2 + m^2 + n^2) = x^2 + y^2 + z^2 = r^2 \Rightarrow l^2 + m^2 + n^2 = 1$

or, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, or, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Note: If OP = r and the co-ordinates of point P be (x, y, z), then d.c.'s of line OP are x/r, y/r, z/r.

Direction cosines of $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are $\frac{a}{|\mathbf{r}|}, \frac{b}{|\mathbf{r}|}, \frac{c}{|\mathbf{r}|}$.

Since $-1 \le \cos x \le 1$, $\forall x \in R$, hence values of l, m, n are such real numbers which are not less than – 1 and not greater than 1. Hence $\text{d.c.'s} \in [-1, 1]$.

The direction cosines of a line parallel to any co-ordinate axis are equal to the direction cosines of the co-ordinate axis.

The number of lines which are equally inclined to the co-ordinate axes is 4.

If l, m, n are the d.c.'s of a line, then the maximum value of $lmn = \frac{1}{3\sqrt{3}}$.

Important Tips

- The angles α, β, γ are called the direction angles of line AB.
- The d.c.'s of line BA are cos (π– α), cos (π – β) and cos (π – γ) i.e., –cosα, –cosβ, –cosγ.
- Angles α, β, γ are not coplanar.
- α + β + γ is not equal to 360° as these angles do not lie in same plane.
- If P(x, y,z) be a point in space such that $\mathbf{r} = \overrightarrow{OP}$ has d.c.'s l, m, n then

$x = l|\mathbf{r}|, y = m|\mathbf{r}|, z = n|\mathbf{r}|$.

- Projection of a vector **r** on the co-ordinate axes are $l|\mathbf{r}|, m|\mathbf{r}|, n|\mathbf{r}|$.
- $\mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ and $\hat{\mathbf{r}} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$

(2) **Direction ratio**

(i) Three numbers which are proportional to the direction cosines of a line are called the direction ratio of that line. If a, b, c are three numbers proportional to direction cosines l, m, n of a line, then a, b, c are called its direction ratios. They are also called direction numbers or direction components.

Hence by definition, we have $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say) ⇒ l = ak, m = bk, n = ck

$$\Rightarrow l^2 + m^2 + n^2 = (a^2 + b^2 + c^2) = k^2 \Rightarrow k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \ m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \ n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

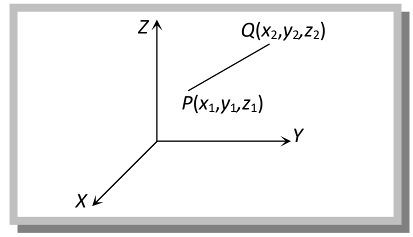
where the sign should be taken all positive or all negative.

Note: Direction ratios are not uniques, whereas d.c.'s are unique. i.e., $a^2 + b^2 + c^2 \neq 1$

(ii) Let $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a vector. Then its d.r.'s are a, b, c

If a vector **r** has d.r.'s a, b, c then $\mathbf{r} = \frac{|\mathbf{r}|}{\sqrt{a^2 + b^2 + c^2}}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$

(iii) **D.c.'s and d.r.'s of a line joining two points:** The direction ratios of line PQ joining $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are $x_2-x_1=a$, $y_2-y_1=b$ and $z_2-z_1=c$ (say).



Then direction cosines are,

$$l=\frac{(x_2-x_1)}{\sqrt{\Sigma(x_2-x_1)^2}}, m=\frac{(y_2-y_1)}{\sqrt{\Sigma(x_2-x_1)^2}}, n=\frac{(z_2-z_1)}{\sqrt{\Sigma(x_2-x_1)^2}}$$

i.e., $l=\frac{x_2-x_1}{PQ}, m=\frac{y_2-y_1}{PQ}, n=\frac{z_2-z_1}{PQ}$.