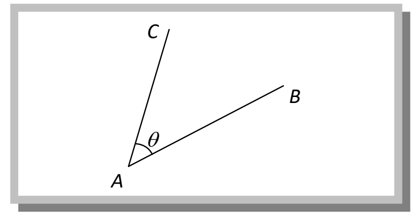
Angle between Two lines.

(1) **Cartesian form:** Let θ be the angle between two straight lines AB and AC whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, is given by



$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

If direction ratios of two lines a_1, b_1, c_1 and a_2, b_2, c_2 are given, then angle between two lines is given by $\cos\theta = \dfrac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

Particular results: We have,

$$\sin^2\theta = 1 - \cos^2\theta = (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$

$$= (l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2$$

$\Rightarrow \sin\theta = \pm\sqrt{\Sigma(l_1 m_2 - l_2 m_1)^2}$, which is known as Lagrange's identity.

The value of sinθ can easily be obtained by the following form.

$$\sin\theta = \sqrt{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ n_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2}$$

When d.r.'s of the lines are given if a_1, b_1, c_1 and a_2, b_2, c_2 are d.r.'s of given two lines, then angle θ between them is given by $\sin\theta = \dfrac{\sqrt{\Sigma(a_1 b_2 - a_2 b_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Condition of perpendicularity: If the given lines are perpendicular, then $\theta = 90°$ i.e. cosθ = 0

$\Rightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ or $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

Condition of parallelism: If the given lines are parallel, then $\theta = 0^o$ i.e. sinθ = 0

$\Rightarrow (l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 = 0$, which is true, only when

$l_1 m_2 - l_2 m_1 = 0$, $m_1 n_2 - m_2 n_1 = 0$ and $n_1 l_2 - n_2 l_1 = 0$

$\Rightarrow \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

Similarly, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Note: The angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

The angle between a diagonal of a cube and the diagonal of a faces of the cube is $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$.

If a straight line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

If the edges of a rectangular parallelopiped be a, b, c, then the angles between the two diagonals are

$$\cos^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right]$$

(2) **Vector form:** Let the vector equations of two lines be $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda\mathbf{b}_2$

As the lines are parallel to the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ respectively, therefore angle between the lines is same as the angle between the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. Thus if θ is the angle between the given lines, then $\cos\theta = \frac{\mathbf{b}_1.\mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$.

Note: If the lines are perpendicular, then $\mathbf{b}_1.\mathbf{b}_2 = 0$.

If the lines are parallel, then $\mathbf{b}_1$ and $\mathbf{b}_2$ are parallel, therefore $\mathbf{b}_1 = \lambda\mathbf{b}_2$ for some scalar λ