Properties of Inverse Trigonometric Functions.

(1) **Meaning of inverse function**

(i) $\sin\theta = x \Rightarrow \sin^{-1} x = \theta$

(ii) $\cos\theta = x \Rightarrow \cos^{-1} x = \theta$

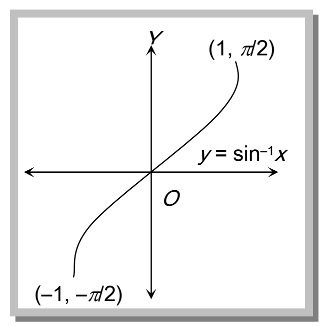
(iii) $\tan\theta = x \Rightarrow \tan^{-1} x = \theta$

(iv) $\cot\theta = x \Rightarrow \cot^{-1} x = \theta$

(v) $\sec\theta = x \Rightarrow \sec^{-1} x = \theta$

(vi) $\text{cosec}\,\theta = x \Rightarrow \text{cosec}^{-1} x = \theta$

(2) **Domain and range of inverse functions**



(i) If $\sin y = x$, then $y = \sin^{-1} x$, under certain condition.

$-1 \le \sin y \le 1$; but $\sin y = x$. $\therefore -1 \le x \le 1$

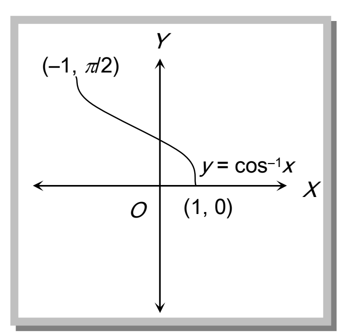
Again, $\sin y = -1 \Rightarrow y = -\frac{\pi}{2}$ and $\sin y = 1 \Rightarrow y = \frac{\pi}{2}$.

Keeping in mind numerically smallest angles or real numbers. $\therefore -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

These restrictions on the values of x and y provide us with the domain and range for the function $y = \sin^{-1} x$.

i.e., Domain : $x \in [-1, 1]$

Range: $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



(ii) Let $\cos y = x$, then $y = \cos^{-1} x$, under certain conditions $-1 \le \cos y \le 1$

$\Rightarrow \ -1 \le x \le 1$

$\cos y = -1 \Rightarrow y = \pi$

$\cos y = 1 \Rightarrow y = 0$

$\therefore 0 \le y \le \pi$ {as cos x is a decreasing function in $[0, \pi]$;

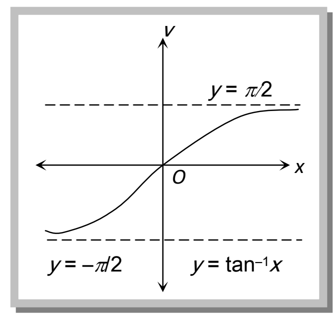
Hence $\cos \pi \le \cos y \le \cos 0$

These restrictions on the values of x and y provide us the domain and range for the function $y = \cos^{-1} x$.

i.e. Domain: $x \in [-1, 1]$

Range: $y \in [0, \pi]$

(iii) If $\tan y = x$, then $y = \tan^{-1} x$, under certain conditions.

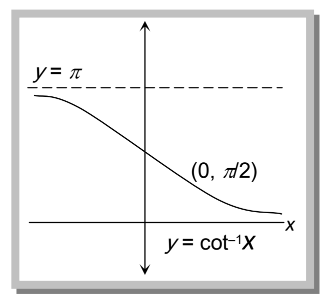


Here, $\tan y \in R \Rightarrow x \in R$, $-\infty < \tan y < \infty \Rightarrow -\frac{\pi}{2} < y < \frac{\pi}{2}$

Thus, Domain $x \in R$;

Range $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) If $\cot y = x$, then $y = \cot^{-1} x$



Under certain conditions, $\cot y \in R \Rightarrow x \in R$;

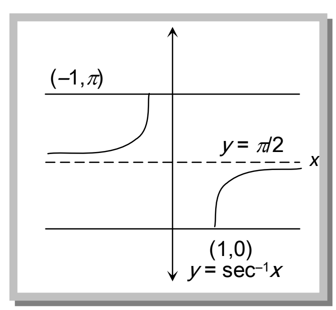
$-\infty < \cot y < \infty \Rightarrow 0 < y < \pi$

These conditions on x and y make the function, $\cot y = x$ one-one and onto so that the inverse function exists. i.e., $y = \cot^{-1} x$ is meaningful.

$\Rightarrow$ Domain : $x \in R$

Range : $y \in (0, \pi)$

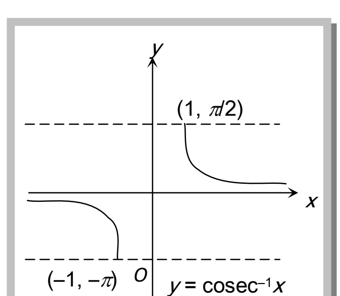
(v) If $\sec y = x$, then $y = \sec^{-1} x$, where $|x| \ge 1$ and $0 \le y \le \pi, y \ne \frac{\pi}{2}$



Here, Domain: $x \in R - (-1, 1)$

Range: $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(vi) If $\text{cosec } y = x$, then $y = \text{cosec}^{-1} x$



Where $|x| \geq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

Here, Domain $\in R-(-1,1)$

Range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Function	**Domain (D)**	**Range (R)**
$\sin^{-1} x$	$-1 \leq x \leq 1$ or $[-1,1]$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$-1 \leq x \leq 1$ or $[-1,1]$	$0 \leq \theta \leq \pi$ or $[0, \pi]$
$\tan^{-1} x$	$-\infty < x < \infty$ i.e., $x \in R$ or $(-\infty, \infty)$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$-\infty < x < \infty$ i.e., $x \in R$ or $(-\infty, \infty)$	$0 < \theta < \pi$ or $(0, \pi)$
$\sec^{-1} x$	$x \leq -1, x \geq 1$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \neq \frac{\pi}{2}, 0 \leq \theta \leq \pi$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\operatorname{cosec}^{-1} x$	$x \leq -1, x \geq 1$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \neq 0, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

(3) $\sin^{-1}(\sin\theta) = \theta$, Provided that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$,

$\cos^{-1}(\cos\theta) = \theta$, Provided that $0 \leq \theta \leq \pi$

$\tan^{-1}(\tan\theta) = \theta$, Provided that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,

$\cot^{-1}(\cot\theta) = \theta$, Provided that $0 < \theta < \pi$

$\sec^{-1}(\sec\theta) = \theta$, Provided that $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$

$\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, Provided that $-\frac{\pi}{2} \leq \theta < 0$ or $0 < \theta \leq \frac{\pi}{2}$

(4) $\sin(\sin^{-1} x) = x$, Provided that $-1 \leq x \leq 1$,

$\cos(\cos^{-1} x) = x$, Provided that $-1 \leq x \leq 1$

$\tan(\tan^{-1} x) = x$, Provided that $-\infty < x < \infty$

$\cot(\cot^{-1} x) = x$, Provided that $-\infty < x < \infty$

$\sec(\sec^{-1} x) = x$, Provided that $-\infty < x \leq -1$ or $1 \leq x < \infty$

$\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, Provided that $-\infty < x \leq -1$ or $1 \leq x < \infty$

(5) $\sin^{-1}(-x) = -\sin^{-1} x$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$,

$\tan^{-1}(-x) = -\tan^{-1} x$

$\cot^{-1}(-x) = \pi - \cot^{-1} x$

$\sec^{-1}(-x) = \pi - \sec^{-1} x$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

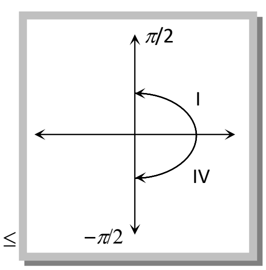
(6) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for all $x \in [-1, 1]$

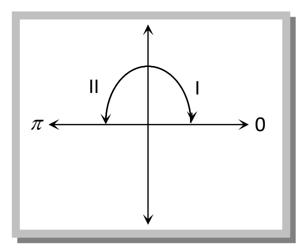
$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, for all $x \in R$

$\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Important Tips

- Here; $\sin^{-1} x, \text{cosec}^{-1} x, \tan^{-1} x$ belong to I and IV Quadrant.
- Here; $\cos^{-1} x, \sec^{-1} x, \cot^{-1} x$ belong to I and II Quadrant.
- I Quadrant is common to all the inverse functions.
- III Quadrant is not used in inverse function.
- IV Quadrant is used in the clockwise direction i.e., $-\frac{\pi}{2} \le y \le$





(7) **Principal values for inverse circular functions**

Principal values for $x \ge 0$	**Principal values for** $x < 0$
$0 \le \sin^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \sin^{-1} x < 0$
$0 \le \cos^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \le \pi$

$0 \le \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 < \cot^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \le \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \le \pi$
$0 < \operatorname{cosec}^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \operatorname{cosec}^{-1} x < 0$

Thus $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, not $\frac{5\pi}{6}$; $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ not $\frac{4\pi}{3}$; $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ not $\frac{2\pi}{3}$;

$\cot^{-1}(-1) = \frac{3\pi}{4}$ not $-\frac{\pi}{4}$ etc.

Note: $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ are also written as arc $\sin x$, arc $\cos x$ and arc $\tan x$ respectively.

It should be noted that if not otherwise stated only principal values of inverse circular functions are to be considered.

(8) **Conversion property :** Let, $\sin^{-1} x = y \Rightarrow x = \sin y \Rightarrow \operatorname{cosec} y = \left(\frac{1}{x}\right) \Rightarrow y = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

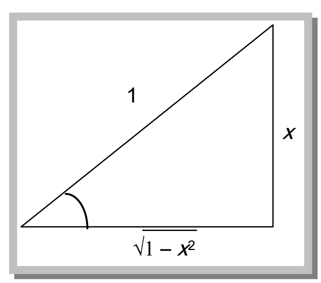
$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Note: $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$$



(9) **General values of inverse circular functions:** We know that if · · · is the smallest angle whose sine is x, then all the angles whose sine is x can be written as $nx + (-1)^n \alpha,$ where $n = 0, 1, 2, ...$ Therefore, the general value of $\sin^{-1} x$ can be taken as $n\pi + (-1)^n \alpha$. The general value of $\sin^{-1} x$ is denoted by $\sin^{-1} x$.

Thus, we have $\boxed{\sin^{-1} x = n\pi + (-1)^n \alpha, -1 \le x \le 1, \text{if} \sin\alpha = x \text{ and } -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}}$

Similarly, general values of other inverse circular functions are given as follows:

$\cos^{-1} x = 2n\pi \pm \alpha, -1 \le x \le 1$;	If $\cos\alpha = x$, $0 \le \alpha \le \pi$
$\tan^{-1} x = n\pi + \alpha,\ x \in R$;	If $\tan\alpha = x,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$
$\cot^{-1} x = n\pi + \alpha,\ x \in R$;	If $\cot\alpha = x$, $0 < \alpha < \pi$
$\sec^{-1} x = 2n\pi \pm \alpha$, $x \ge 1$ or $x \le -1$;	If $\sec\alpha = x, 0 \le \alpha \le \pi$ and $\ne \frac{\pi}{2}$
$\text{cosec}^{-1} x = n\pi + (-1)^n \alpha,\ x \ge 1$ or $x \le -1$;	If $\text{cosec}\,\alpha = x,\ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ and $x \ne 0$