Separation of Inverse Trigonometric and Inverse Hyperbolic Functions.

If $\sin(\alpha + i\beta) = x + iy$ then $(\alpha + i\beta)$, is called the inverse sine of $(x + iy)$. We can write it as,

$\sin^{-1}(x + iy) = \alpha + i\beta$

Here the following results for inverse functions may be easily established.

(1)

$$\cos^{-1}(x+iy) = \frac{1}{2}\cos^{-1}\left[(x^2+y^2) - \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right] + \frac{i}{2}\cosh^{-1}\left[(x^2+y^2) + \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right]$$

(2) $\sin^{-1}(x+iy) = \frac{\pi}{2} - \cos^{-1}(x+iy)$

=

$$\frac{\pi}{2} - \frac{1}{2}\cos^{-1}\left[(x^2+y^2) - \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right] - \frac{i}{2}\cosh^{-1}\left[(x^2+y^2) + \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right]$$

(3) $\tan^{-1}(x+iy) = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) + \frac{i}{2}\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right) =$

$$\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) + \frac{i}{4}\log\left[\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right]$$

(4) $\sin^{-1}(\cos\theta + i\sin\theta) = \cos^{-1}(\sqrt{\sin\theta}) + i\sinh^{-1}(\sqrt{\sin\theta})$ or $\cos^{-1}(\sqrt{\sin\theta}) + i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(5) $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}(\sqrt{\sin\theta}) - i\sinh^{-1}(\sqrt{\sin\theta})$ or $\sin^{-1}(\sqrt{\sin\theta}) - i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(6) $\tan^{-1}(\cos\theta + i\sin\theta) = \frac{\pi}{4} + \frac{i}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right), (\cos\theta) > 0$

and $\tan^{-1}(\cos\theta + i\sin\theta) = \left(-\frac{\pi}{4}\right) + \frac{1}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right), (\cos\theta) < 0$

Since each inverse hyperbolic function can be expressed in terms of logarithmic function, therefore for separation into real and imaginary parts of inverse hyperbolic function of complex quantities use the appropriate method.

Note: Both inverse circular and inverse hyperbolic functions are many valued.