Relation between Hyperbolic and Circular Functions.

We have from Euler formulae,

$e^{ix} = \cos x + i \sin x$ ....(i) and $e^{-ix} = \cos x - i \sin x$ ....(ii)

Adding (i) and (ii) $\Rightarrow \cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

Subtracting (ii) from (i) $\Rightarrow \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Replacing x by ix in these values, we get $\cos(ix) = \dfrac{e^{-x} + e^{x}}{2} = \cosh x$

$\therefore \cos(ix) = \cosh x$

$$\sin(ix) = \frac{e^{-x} - e^{x}}{2i} = i\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$\therefore \sin(ix) = i \sinh x$

Also $\tan(ix) = \dfrac{\sin(ix)}{\cos(ix)} = \dfrac{i \sinh x}{\cosh x}$

$\tan(ix) = i \tanh x$

Similarly replacing x by ix in the definitions of $\sinh x$ and $\cosh x$, we get

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

Also, $\tanh(ix) = \dfrac{\sinh(ix)}{\cosh(ix)} = \dfrac{i \sin x}{\cos x} = i \tan x$

Thus, we obtain the following relations between hyperbolic and trigonometrical functions.

(1) $\sin(ix) = i \sinh x$	(2) $\cos(ix) = \cosh x$
$\sinh(ix) = i \sin x$	$\cosh(ix) = \cos x$
$\sinh x = -i \sin(ix)$	$\cosh x = \cos(ix)$
$\sin x = -i \sin h(ix)$	$\cos x = \cos h(ix)$

(3) $\tan(ix) = i\tanh x$ $\tanh(ix) = i\tan x$ $\tanh x = -i\tan(ix)$ $\tan x = -i\tanh(ix)$	(4) $\cot(ix) = -i\coth x$ $\coth(ix) = -i\cot x$ $\coth x = i\cot(ix)$ $\cot x = i\coth(ix)$
(5) $\sec(ix) = \text{sech} x$ $\sec \text{h}(ix) = \sec x$ $\sec \text{h}x = \sec(ix)$ $\sec x = \text{sech}(ix)$	(6) $cosec(ix) = -i\,\text{cosech}\, x$ $\text{cosech}\,(ix) = i\,\text{cosec}\, x$ $\text{cosech}\, x = i\,\text{cosec}\,(ix)$ $cosecx = i\,\text{cosech}\,(ix)$

Important Tips

☞ For obtaining any formula given in (5)th article, use the following substitutions in the corresponding formula for trigonometric functions.

$\sin x \longrightarrow i\sinh x$ $\quad$ $\cos x \longrightarrow \cosh x$ $\quad$ $\tan x \longrightarrow i\tanh x$

$\sin^2 x \longrightarrow -\sinh^2 x$ $\quad$ $\cos^2 x \longrightarrow \cosh^2 x$ $\quad$ $\tan^2 x \longrightarrow -\tanh^2 x$

For example,

For finding out the formula for $\cosh 2x$ in terms of $\tanh x$, replace $\tan x$ by $i\tanh x$ and $\tan^2 x$ by $\tan^2 x$ by $-\tanh^2 x$ in the following formula of trigonometric function of $\cos 2x$:

$\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$ we get, $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$