Inverse Hyperbolic Functions.

If $\sinh y = x$, then y is called the inverse hyperbolic sine of x and it is written as $y = \sinh^{-1} x$.
Similarly $cosech^{-1} x, \cosh^{-1} x, \tanh^{-1} x$ etc. can be defined.

(1) **Domain and range of Inverse hyperbolic function**

Function	**Domain**	**Range**
$\sinh^{-1} x$	R	R
$\cosh^{-1} x$	[1, ∞)	R
$\tanh^{-1} x$	$(-1, 1)$	R
$\coth^{-1} x$	R – [–1, 1]	R_0
$\text{sech}^{-1} x$	(0, 1]	R
$\text{cosech}^{-1} x$	R_0	R_0

(2) **Relation between inverse hyperbolic function and inverse circular function**

Method: Let $\sinh^{-1} x = y$

$\Rightarrow \quad x = \sinh y = -i \sin(iy) \Rightarrow ix = \sin(iy) \Rightarrow iy = \sin^{-1}(ix)$

$\Rightarrow y = -i \sin^{-1}(ix) \Rightarrow \sinh^{-1} x = -i \sin^{-1}(ix)$

Therefore we get the following relations

(i) $\sinh^{-1} x = -i \sin^{-1}(ix)$

(ii) $\cosh^{-1} x = -i \cos^{-1} x$

(iii) $\tanh^{-1} x = -i \tan^{-1}(ix)$

(iv) $\sec h^{-1} x = -i \sec^{-1} x$

(v) $cosech^{-1} x = i \,\text{cosec}^{-1}(ix)$

(3) **To express any one inverse hyperbolic function in terms of the other inverse hyperbolic functions**

To express $\sinh^{-1} x$ in terms of the others

(i) Let $\sinh^{-1} x = y \Rightarrow x = \sinh y \Rightarrow \text{cosech}\, y = \frac{1}{x} \Rightarrow y = cosec^{-1}\left(\frac{1}{x}\right)$

(ii) $\because \cosh y = \sqrt{1+\sinh^2 y} = \sqrt{1+x^2}$

$\because y = \cosh^{-1}\sqrt{1+x^2} \Rightarrow \sinh^{-1} x = \cosh^{-1}\sqrt{1+x^2}$

(iii) $\because \tanh y = \frac{\sinh y}{\cosh y} = \frac{\sinh y}{\sqrt{1+\sinh^2 y}} = \frac{x}{\sqrt{1+x^2}}$

$\therefore y = \tanh^{-1}\frac{x}{\sqrt{1+x^2}} \Rightarrow \sinh^{-1} x = \tanh^{-1}\frac{x}{\sqrt{1+x^2}}$

(iv) $\because \coth y = \frac{\sqrt{1+\sinh^2 y}}{\sinh y} = \frac{\sqrt{1+x^2}}{x}$

$\therefore y = \coth^{-1}\frac{\sqrt{1+x^2}}{x} \quad \Rightarrow \sinh^{-1} x = \coth^{-1}\frac{\sqrt{1+x^2}}{x}$

(v) $\because \text{sec h} y = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$

$y = \text{sec h}^{-1}\frac{1}{\sqrt{1+x^2}} \Rightarrow \sinh^{-1} x = \text{sec h}^{-1}\frac{1}{\sqrt{1+x^2}}$

(vi) Also, $\sinh^{-1} x = \text{cosech}^{-1}\left(\frac{1}{x}\right)$

From the above, it is clear that

$\coth^{-1} x = \tanh^{-1}\left(\frac{1}{x}\right)$

$\text{sec h}^{-1} x = \cosh^{-1}\left(\frac{1}{x}\right)$

$\text{cosech}^{-1} = \sinh^{-1}\left(\frac{1}{x}\right)$

Note: If x is real then all the above six inverse functions are single valued.

(4) **Relation between inverse hyperbolic functions and logarithmic functions**

Method:

Let $\sinh^{-1} x = y$

$\Rightarrow x = \sinh y = \dfrac{e^y - e^{-y}}{2} \Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = \dfrac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$

But $e^y > 0, \forall y$ and $x < \sqrt{x^2 + 1}$

$\therefore e^y = x + \sqrt{x^2 + 1} \Rightarrow y = \log(x + \sqrt{x^2 + 1})$

$\therefore \sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$

By the above method we can obtain the following relations between inverse hyperbolic functions and principal values of logarithmic functions.

(i) $\sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$ $\quad (-\infty < x < \infty)$

(ii) $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$ $\quad (x \geq 1)$

(iii) $\tanh^{-1} x = \dfrac{1}{2}\log\left(\dfrac{1 + x}{1 - x}\right)$ $\quad |x| < 1$

(iv) $\coth^{-1} x = \dfrac{1}{2}\log\left(\dfrac{x + 1}{x - 1}\right)$ $\quad |x| > 1$

(v) $\operatorname{sech}^{-1} x = \log\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$ $\quad 0 < x \leq 1$

(vi) $\operatorname{cosech}^{-1} x = \log\left(\dfrac{1 + \sqrt{1 + x^2}}{x}\right)$ $\quad (x \neq 0)$

Note: Formulae for values of $\operatorname{cosech}^{-1} x$, $\operatorname{sech}^{-1} x$ and $\coth^{-1} x$ may be obtained by replacing x by $\dfrac{1}{x}$ in the values of $\sinh^{-1} x, \cosh^{-1} x$ and $\tanh^{-1} x$ respectively.