Addition Theorems on Probability.

Notations: (i) $P(A+B)$ or $P(A\cup B)$ = Probability of happening of A or B

= Probability of happening of the events A or B or both

= Probability of occurrence of at least one event A or B

(ii) P(AB) or P(A∩B) = Probability of happening of events A and B together.

(1) **When events are not mutually exclusive:** If A and B are two events which are not mutually exclusive, then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ or $P(A+B)=P(A)+P(B)-P(AB)$.

For any three events A, B, C

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$

or $P(A+B+C)=P(A)+P(B)+P(C)-P(AB)-P(BC)-P(CA)+P(ABC)$.

(2) **When events are mutually exclusive:** If A and B are mutually exclusive events, then

$n(A\cap B)=0\Rightarrow P(A\cap B)=0$

$\therefore P(A\cup B)=P(A)+P(B)$.

For any three events A, B, C which are mutually exclusive,

$P(A\cap B)=P(B\cap C)=P(C\cap A)=P(A\cap B\cap C)=0$ $\therefore P(A\cup B\cup C)=P(A)+P(B)+P(C)$.

The probability of happening of any one of several mutually exclusive events is equal to the sum of their probabilities, i.e. if $A_1, A_2 \ldots\ldots A_n$ are mutually exclusive events, then

$P(A_1+A_2+\ldots+A_n)=P(A_1)+P(A_2)+\ldots\ldots+P(A_n)$ i.e. $P(\sum A_i)=\sum P(A_i)$.

(3) **When events are independent:** If A and B are independent events, then

$P(A\cap B)=P(A).P(B)$

$\therefore P(A\cup B)=P(A)+P(B)-P(A).P(B)$.

(4) **Some other theorems**

(i) Let A and B be two events associated with a random experiment, then

(a) $P(\overline{A}\cap B)=P(B)-P(A\cap B)$ (b) $P(A\cap\overline{B})=P(A)-P(A\cap B)$

If B ⊂ A, then

(a) $P(A \cap \overline{B}) = P(A) - P(B)$ (b) $P(B) \le P(A)$

Similarly if A ⊂ B, then

(a) $(\overline{A} \cap B) = P(B) - P(A)$ (b) $P(A) \le P(B)$.

Note: Probability of occurrence of neither A nor B is $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$.

(ii) **Generalization of the addition theorem:** If $A_1, A_2, \ldots\ldots, A_n$ are n events associated with a random experiment, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1 \\ i \ne j}}^{n} P(A_i \cap A_j) + \sum_{\substack{i,j,k=1 \\ i \ne j \ne k}}^{n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots\ldots \cap A_n).$$

If all the events $A_i (i = 1, 2 \ldots, n)$ are mutually exclusive, then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

i.e. $P(A_1 \cup A_2 \cup \ldots . \cup A_n) = P(A_1) + P(A_2) + \ldots . + P(A_n)$.

(iii) **Booley'sinequality:**If $A_1, A_2, \ldots . A_n$ are n events associated with a random experiment, then

(a) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$ (b) $P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$

These results can be easily established by using the Principle of Mathematical Induction.

Important Tips

Let A, B, and C are three arbitrary events. Then

Verbal description of event	**Equivalent Set Theoretic Notation**
(i) Only A occurs	(i) $A \cap \overline{B} \cap \overline{C}$
(ii) Both A and B, but not C occur	(ii) $A \cap B \cap \overline{C}$
(iii) All the three events occur	(iii) $A \cap B \cap C$
(iv) At least one occurs	(iv) $A \cup B \cup C$
(v) At least two occur	(v) $(A \cap B) \cup (B \cap C) \cup (A \cap C)$

(vi) One and no more occurs	(vi) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$
(vii) Exactly two of A, B and C occur	(vii) $(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$
(viii) None occurs	(viii) $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$
(ix) Not more than two occur	(ix) $(A \cap B) \cup (B \cap C) \cup (A \cap C) - (A \cap B \cap C)$
(x) Exactly one of A and B occurs	(x) $(A \cap \overline{B}) \cup (\overline{A} \cap B)$