## Solution of Algebraic and Transcendental Equations.

There are many numerical methods for solving algebraic and transcendental equations. Some of these methods are given below. After locating root of an equation, we successively approximate it to any desired degree of accuracy.
(1) Iterative method: If the equation $\mathrm{f}(\mathrm{x})=0$ can be expressed as $\mathrm{x}=\mathrm{g}(\mathrm{x})$ (certainly $\mathrm{g}(\mathrm{x})$ is nonconstant), the value $\mathrm{g}\left(\mathrm{x}_{0}\right)$ of $\mathrm{g}(\mathrm{x})$ at $x=x_{0}$ is the next approximation to the root $\alpha$. Let $g\left(x_{0}\right)=x_{1}$, then $x_{2}=g\left(x_{1}\right)$ is a third approximation to $\alpha$. This process is repeated until a number, whose absolute difference from $\alpha$ is as small as we please, is obtained. This number is the required root of $f(x)=0$, calculated upto a desired accuracy.

Thus, if $x_{i}$ is an approximation to $\alpha$, then the next approximation $x_{i+1}=g\left(x_{i}\right)$
The relation (i) is known as Iterative formula or recursion formula and this method of approximating a real root of an equation $f(x)=0$ is called iterative method.
(2) Successive bisection method:This method consists in locating the root of the equation $f(x)=0$ between a and b . If $\mathrm{f}(\mathrm{x})$ is continuous between a and $b$, and $f(a)$ and $f(b)$ are of opposite signs i.e. $f(a) . f(b)<0$, then there is
 (at least one) root between $a$ and $b$. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root $x_{1}=\frac{1}{2}(a+b)$.

## Working Rule:

(i) Find $f(a)$ by the above formula.
(ii) Let $f(a)$ be negative and $f(b)$ be positive, then take $x_{1}=\frac{a+b}{2}$.
(iii) If $f\left(x_{1}\right)=0$, then c is the required root or otherwise if $f\left(x_{1}\right)$ is negative then root will be in $\left(x_{1}, b\right)$ and if $f\left(x_{1}\right)$ is positive then root will be in $\left(a, x_{1}\right)$.
(iv) Repeat it until you get the root nearest to the actual root.

Note: This method of approximation is very slow but it is reliable and can be applied to any type of algebraic or transcendental equations.

This method may give a false root if $f(x)$ is discontinuous on $[\mathrm{a}, \mathrm{b}]$.
(3) Method of false position or Regula-Falsi method:This is the oldest method of finding the real root of an equation $f(x)=0$ and closely resembles the bisection method. Here we choose two points $x_{0}$ and $x_{1}$ such that $f\left(x_{0}\right)$ and $f\left(x_{1}\right)$ are of opposite signs i.e. the graph of $y=f(x)$ crosses the $x$-axis between these points. This indicates that a root lies between $x_{0}$ and $x_{1}$ consequently $f\left(x_{0}\right) f\left(x_{1}\right)<0$.


Equation of the chord joining the points $A\left[x_{0}, f\left(x_{0}\right)\right]$ and $B\left[x_{1}, f\left(x_{1}\right)\right]$ is
$y-f\left(x_{0}\right)=\frac{f\left(x_{1}\right)-f\left(x_{0}\right)}{x_{1}-x_{0}}\left(x-x_{0}\right)$
The method consists in replacing the curve $A B$ by means of the chord $A B$ and taking the point of intersection of the chord with the $x$-axis as an approximation to the root. So the abscissa of the point where the chord cuts the x -axis $(y=0)$ is given by $x_{2}=x_{0}-\frac{x_{1}-x_{0}}{f\left(x_{1}\right)-f\left(x_{2}\right)} f\left(x_{0}\right)$
Which is an approximation to the root.
If now $f\left(x_{0}\right)$ and $f\left(x_{2}\right)$ are of opposite signs, then the root lies between $x_{0}$ and $x_{2}$. So replacing $x_{1}$ by $x_{2}$ in (ii), we obtain the next approximation $x_{3}$. (The root could as well lie between $x_{1}$ and $x_{2}$ and we
would obtain $x_{3}$ accordingly). This procedure is repeated till the root is found to desired accuracy. The iteration process based on (i) is known as the method of false position.

## Working rule

(i) Calculate $f\left(x_{0}\right)$ and $f\left(x_{1}\right)$, if these are of opposite sign then the root lies between $x_{0}$ and $x_{1}$.
(ii) Calculate $x_{2}$ by the above formula.
(iii) Now if $f\left(x_{2}\right)=0$, then $x_{2}$ is the required root.
(iv) If $f\left(x_{2}\right)$ is negative, then the root lies in $\left(x_{2}, x_{1}\right)$.
(v) If $f\left(x_{2}\right)$ is positive, then the root lies in $\left(x_{0}, x_{2}\right)$.
(vi) Repeat it until you get the root nearest to the real root.

Note: This method is also known as the method of false position.
The method may give a false root or may not converge if either a and b are not sufficiently close to each other or $f(x)$ is discontinuous on $[\mathrm{a}, \mathrm{b}]$.

Geometrically speaking, in this method, part of the curve between the points $P(a, f(a))$ and $Q(b, f(b))$ is replaced by the secant PQ and the point of intersection of this secant with x -axis gives an approximate value of the root.

It converges more rapidly than bisection.
(4) Newton-Raphson method:Let $x_{0}$ be an approximate root of the equation $f(x)=0$. If $x_{1}=x_{0}+h$ be the exact root, then $f\left(x_{1}\right)=0$
$\therefore$ Expanding $f\left(x_{0}+h\right)$ by Taylor's series
$f\left(x_{0}\right)+h f^{\prime}\left(x_{0}\right)+\frac{h^{2}}{2!} f^{\prime}\left(x_{0}\right)+\ldots \ldots . .=0$
Since h is small, neglecting $h^{2}$ and higher powers of $h$, we get
$f\left(x_{0}\right)+h f^{\prime}\left(x_{0}\right)=0$
or $\quad h=-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}$
$\therefore$ A closer approximation to the root is given by
$x_{1}=x_{0}-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}$
Similarly, starting with $x_{1}$, a still better approximation $x_{2}$ is given by
$x_{2}=x_{1}-\frac{f\left(x_{1}\right)}{f^{\prime}\left(x_{1}\right)}$
In general, $\quad x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}$
Which is known as the Newton-Raphson formula or Newton's iteration formula.

## Working rule:

(i) Find $|f(a)|$ and $|f(b)|$. If $|f(a)|<|f(b)|$, then let $a=x_{0}$, otherwise $b=x_{0}$.
(ii) Calculate $x_{1}=x_{0}-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}$
(iii) $x_{1}$ is the required root if $f\left(x_{1}\right)=0$.
(iv) To find nearest to the real root, repeat it.

Note: Geometrically speaking, in Newton-Raphson method, the part of the graph of the function $y=f(x)$ between the point $P(a, f(a))$ and the $x$-axis is replaced by a tangent to the curve at the point at each step in the approximation process.

- This method is very useful for approximating isolated roots.
- The Newton-Raphson method fails if $f^{\prime}(x)$ is difficult to compute or vanishes in a neighbourhood of the desired root. In such cases, the Regula-Falsi method should be used.
- The Newton-Raphson method is widely used since in a neighbourhood of the desired root, it converges more rapidly than the bisection method or the Regula-Falsi method.
- If the starting value a is not close enough to the desired root, the method may give a false root or may not converge.
- If $f\left(x_{0}\right) / f^{\prime}\left(x_{0}\right)$ is not sufficiently small, this method does not work. Also if it work, it works faster.


## Geometrical Interpretation

Let $x_{0}$ be a point near the root $\alpha$ of the equation $f(x)=0$. Then the equation of the tangent at $A_{0}\left[x_{0}, f\left(x_{0}\right)\right]$ is $y-f\left(x_{0}\right)=f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)$.
It cuts the x -axis at $x_{1}=x_{0}-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}$


Which is a first approximation to the root $\alpha$. If $A_{1}$ is the point corresponding to $x_{1}$ on the curve, then the tangent at $A_{1}$ will cut the x -axis of $x_{2}$ which is nearer to $\alpha$ and is, therefore, a second approximation to the root. Repeating this process, we approach to the root $\alpha$ quite rapidly. Hence the method consists in replacing the part of the curve between the point $A_{0}$ and the xaxis by means of the tangent to the curve at $A_{0}$.

