Multinomial Theorem.

Let $x_1, x_2, ..., x_m$ be integers. Then number of solutions to the equation

$x_1 + x_2 + ... + x_m = n$ .....(i)

Subject to the condition $a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2, ..., a_m \le x_m \le b_m$ .....(ii)

is equal to the coefficient of x^n in

$(x^{a_1} + x^{a_1+1} + ... + x^{b_1})(x^{a_2} + x^{a_2+1} + ... + x^{b_2})......(x^{a_m} + x^{a_{m+1}} + ... + x^{b_m})$ ......(iii)

This is because the number of ways, in which sum of m integers in (i) equals n, is the same as the number of times x^n comes in (iii).

(1) **Use of solution of linear equation and coefficient of a power in expansions to find the number of ways of distribution:**

(i) the number of integral solutions of $x_1 + x_2 + x_3 + ... + x_r = n$ where $x_1 \ge 0, x_2 \ge 0, ... x_r \ge 0$ is the same as the number of ways to distribute n identical things among r persons.

This is also equal to the coefficient of x^n in the expansion of $(x^0 + x^1 + x^2 + x^3 + ...)^r$

= coefficient of x^n in $\left(\frac{1}{1-x}\right)^r$ = coefficient of x^n in $(1-x)^{-r}$

= coefficient of x^n in $\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + ... + \frac{r(r+1)(r+2)......(r+n-1)}{n!}x^n + ...\right.$

$= \frac{r(r+1)(r+2)....(r+n-1)}{n!} = \frac{(r+n-1)!}{n!(r-1)!} = {}^{n+r-1}C_{r-1}$

(ii) The number of integral solutions of $x_1 + x_2 + x_3 + ... + x_r = n$ where $x_1 \geq 1, x_2 \geq 1, ... x_r \geq 1$ is same as the number of ways to distribute n identical things among r persons each getting at least 1. This also equal to the coefficient of x^n in the expansion of $(x^1 + x^2 + x^3 + ...)^r$

= coefficient of x^n in $\left(\frac{x}{1-x}\right)^r$ = coefficient of x^n in $x^r(1-x)^{-r}$

= coefficient of x^n in $x^r\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + ... + \frac{r(r+1)(r+2).....(r+n-1)}{n!}x^n + ...\right\}$

= coefficient of x^{n-r} in $\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + ... + \frac{r(r+1)(r+2).....(r+n-1)}{n!}x^n + ...\right\}$

= $\frac{r(r+1)(r+2)......(r+n-r-1)}{(n-r)!}$ = $\frac{r(r+1)(r+2).....(n-1)}{(n-r)!}$ = $\frac{(n-1)!}{(n-r)!(r-1)!} = {}^{n-1}C_{r-1}$.