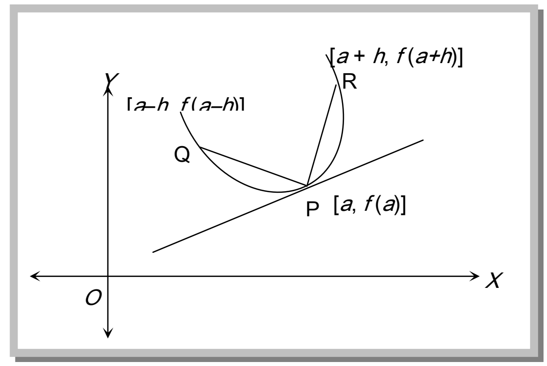
Differentiability of a Function at a Point.

(1) **Meaning of differentiability at a point:**



Consider the function $f(x)$ defined on an open interval (b, c) let $P(a, f(a))$ be a point on the curve $y = f(x)$ and let $Q(a-h, f(a-h))$ and $R(a+h, f(a+h))$ be two neighboring points on the left hand side and right hand side respectively of the point P.

Then slope of chord $PQ = \dfrac{f(a-h)-f(a)}{(a-h)-a} = \dfrac{f(a-h)-f(a)}{-h}$

And, slope of chord $PR = \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$.

$\because$ As $h \to 0$, point *Q* and *R* both tends to *P* from left hand and right hand respectively. Consequently, chords *PQ* and *PR* becomes tangent at point *P*.

Thus, $\lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0}$ (slope of chord *PQ*) = $\lim\limits_{Q\to P}$ (slope of chord *PQ*)

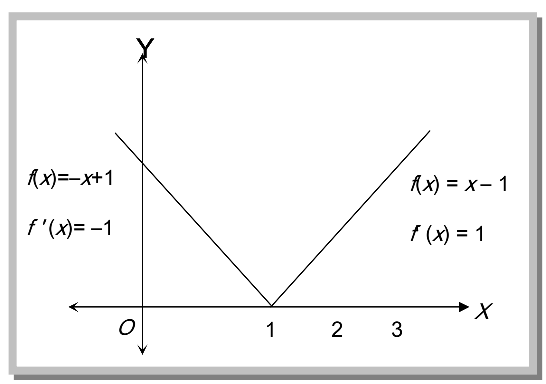
Slope of the tangent at point *P*, which is limiting position of the chords drawn on the left hand side of point *P* and $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ = $\lim\limits_{h\to 0}$(slope of chord *PR*) = $\lim\limits_{R\to P}$ (slope of chord *PR*).

$\Rightarrow$ Slope of the tangent at point *P*, which is the limiting position of the chords drawn on the right hand side of point *P*.

Now, $f(x)$ is differentiable at $x = a \Leftrightarrow \lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

$\Leftrightarrow$ There is a unique tangent at point *P*.

Thus, $f(x)$ is differentiable at point *P*, iff there exists a unique tangent at point *P*. In other words, $f(x)$ is differentiable at a point *P* iff the curve does not have *P* as a corner point. *i.e.*, "the function is not differentiable at those points on which function has jumps (or holes) and sharp edges."



Let us consider the function $f(x) = |x-1|$, which can be graphically shown,

Which show $f(x)$ is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.

Mathematically: The right hand derivative at $x=1$ is 1 and left-hand derivative at $x=1$ is –1. Thus, $f(x)$ is not differentiable at $x=1$.

(2) **Right hand derivative:** Right hand derivative of $f(x)$ at $x=a$, denoted by $f'(a+0)$ or $f'(a+)$, is the $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

(3) **Left hand derivative:** Left hand derivative of $f(x)$ at $x=a,$ denoted by $f'(a-0)$ or $f'(a-)$, is the $\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$.

(4) A function $f(x)$ is said to be differentiable (finitely) at $x = a$ if $f'(a+0)=f'(a-0)$ = finite

i.e., $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$ = finite and the common limit is called the derivative of $f(x)$ at $x=a$, denoted by $f'(a)$. Clearly, $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ {$x\to a$ from the left as well as from the right}.