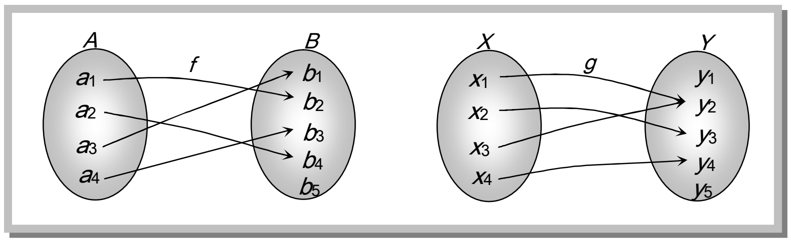
Kinds of Function.

(1) **One-one function (injection):** A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of *A* have different images in *B*. Thus, $f : A \to B$ is one-one.

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$ For all $a, b \in A \Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

E.g. Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams.



Clearly, $f : A \to B$ is a one-one function. But $g : X \to Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function *g*.

(i) **Method to check the injectivity of a function**

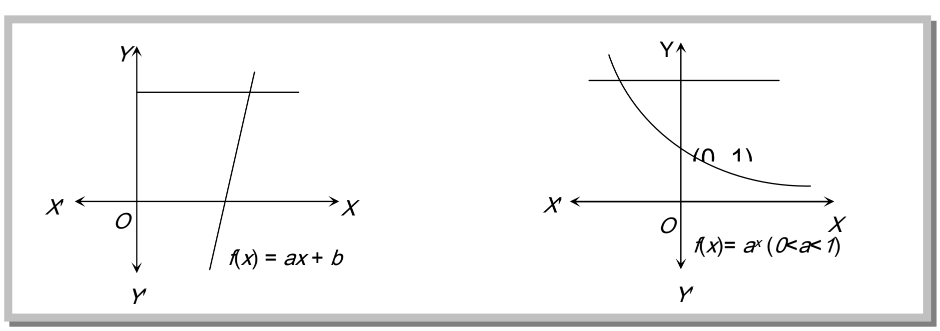
Step I: Take two arbitrary elements *x*, *y* (say) in the domain of *f*.

Step II: Put $f(x) = f(y)$.

Step III: Solve $f(x) = f(y)$. If $f(x) = f(y)$ gives $x = y$ only, then $f : A \to B$ is a one-one function (or an injection). Otherwise not.

Note: If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

If the graph of the function $y = f(x)$ is given and each line parallel to *x*-axis cuts the given curve at maximum one point then function is one-one. *e.g.*

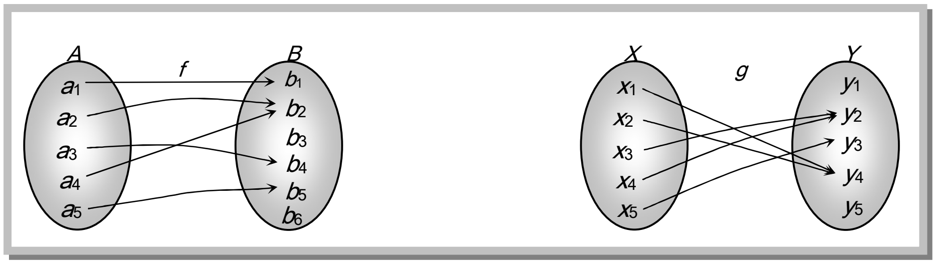


(ii) **Number of one-one functions (injections) :** If A and B are finite sets having m and n elements respectively, then number of one-one functions from A to B $= \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(2) **Many-one function:** A function $f: A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

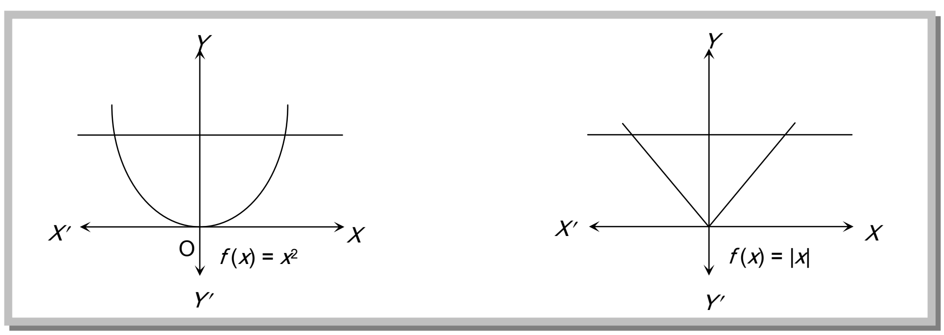
Thus, $f: A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

In other words, $f: A \to B$ is a many-one function if it is not a one-one function.



Note: If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same then function is many-one.

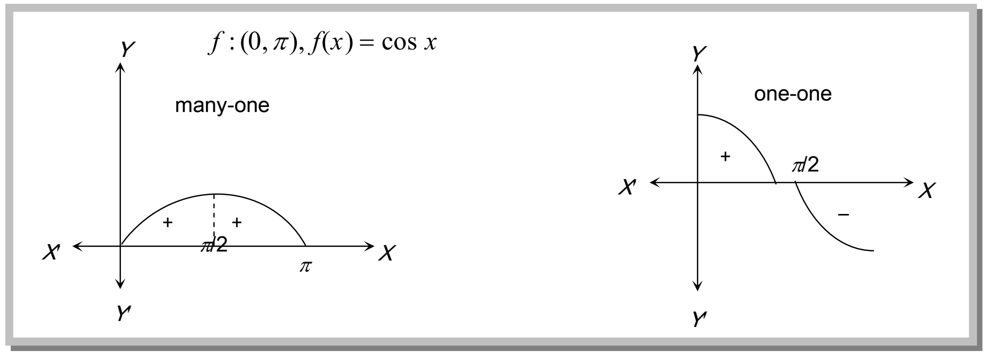
❒ If the graph of $y = f(x)$ is given and the line parallel to x-axis cuts the curve at more than one point then function is many-one.



❒ If the domain of the function is in one quadrant then the trigonometrical functions are always one-one.

❒ If trigonometrical function changes its sign in two consecutive quadrants then it is one-one but if it does not change the sign then it is many-one.

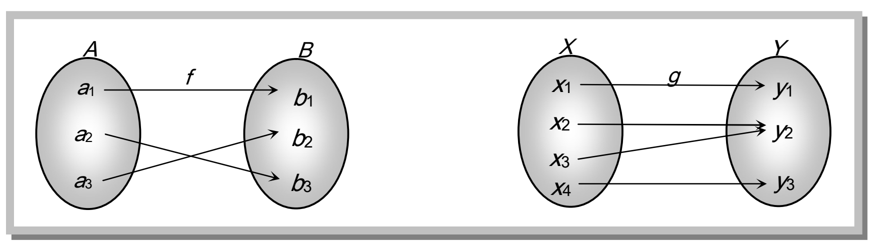
$f:(0,\pi), f(x)=\sin x$



❒ In three consecutive quadrants trigonometrical functions are always many-one.

(3) **Onto function (surjection):**A function $f: A \to B$ is onto if each element of *B* has its pre-image in *A*. Therefore, if $f^{-1}(y) \in A, \ \forall y \in B$ then function is onto. In other words, Range of *f* = Co-domain of *f*.

E.g. The following arrow-diagram shows onto function.

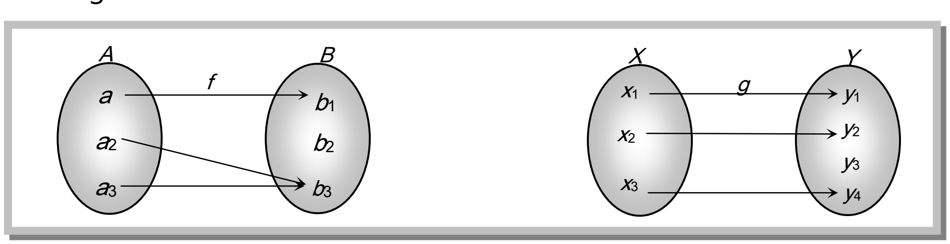


(i) **Number of onto function (surjection) :** If *A* and *B* are two sets having *m* and *n* elements respectively such that $1 \le n \le m$, then number of onto functions from *A* to *B* is $\sum_{r=1}^{n}(-1)^{n-r}\,{}^{n}C_r r^m$.

(4) **Into function:***A* function $f: A \to B$ is an into function if there exists an element in *B* having no pre-image in *A*.

In other words, $f: A \to B$ is an into function if it is not an onto function.

E.g. The following arrow-diagram shows into function.



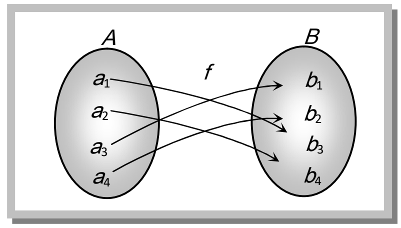
(i) **Method to find onto or into function**

(a) If range = co-domain, then $f(x)$ is onto and if range is a proper subset of the co-domain, then $f(x)$ is into.

(b) Solve $f(x) = y$ by taking x as a function of y *i.e.*, $g(y)$ (say).

(c) Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain for $y \in$ co-domain, then $f(x)$ is onto and if any one of the above requirements is not fulfilled, then $f(x)$ is into.

(5) **One-one onto function (bijection):** A function $f : A \to B$ is a bijection if it is one-one as well as onto.



In other words, a function $f : A \to B$ is a bijection if

(i) It is one-one i.e., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) It is onto i.e., for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Clearly, f is a bijection since it is both injective as well as surjective.

Number of one-one onto function (bijection): If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijection from A to B is the total number of arrangements of n items taken all at a time *i.e.* $n!$.

(6) **Algebraic functions:** Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations +, –, × and ÷ are called algebraic functions.

e.g., (i) $x^{\frac{3}{2}} + 5x$ (ii) $\frac{\sqrt{x+1}}{x-1}, x \neq 1$ (iii) $3x^4 - 5x + 7$

The algebraic functions can be classified as follows:

(i) **Polynomial or integral function:** It is a function of the form $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, Where $a_0 \neq 0$ and $a_0, a_1, ..., a_n$ are constants and $n \in N$ is called a polynomial function of degree n

E.g. $f(x) = x^3 - 2x^2 + x + 3$ is a polynomial function.

Note: The polynomial of first degree is called alinear function and polynomial of zero degree is called a constant function.

(ii) **Rational function:** The quotient of two polynomial functions is called the rational function.

E.g. $f(x) = \dfrac{x^2 - 1}{2x^3 + x^2 + 1}$ is a rational function.

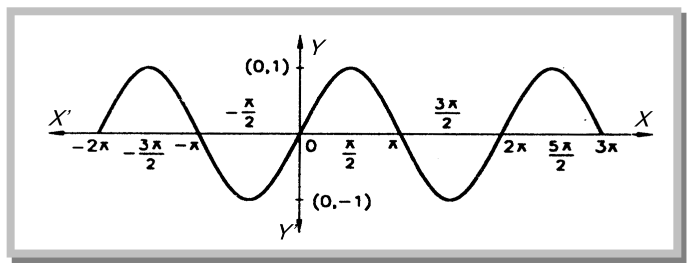
(iii) **Irrational function:** An algebraic function which is not rational is called an irrational function.

E.g. $f(x) = x + \sqrt{x} + 6,\ g(x) = \dfrac{x^3 - \sqrt{x}}{1 + x^{1/4}}$ are irrational functions.

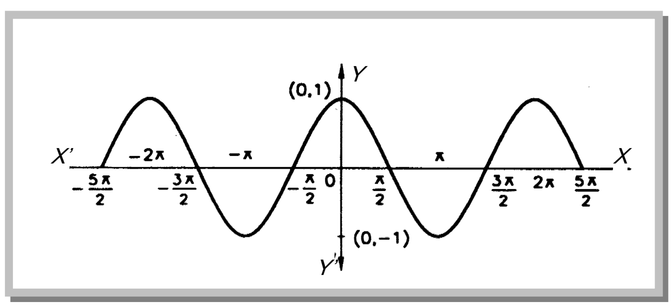
(7) **Transcendental function:** A function which is not algebraic is called a transcendental function. *e.g.*, trigonometric; inverse trigonometric, exponential and logarithmic functions are all transcendental functions.

(i) **Trigonometric functions:** A function is said to be a trigonometric function if it involves circular functions (sine, cosine, tangent, cotangent, secant, and cosecant) of variable angles.

(a) **Sine function:** The function that associates to each real numbers x to $\sin x$ is called the sine function. Here x is the radian measure of the angle. The domain of the sine function is R and the range is [–1, 1].

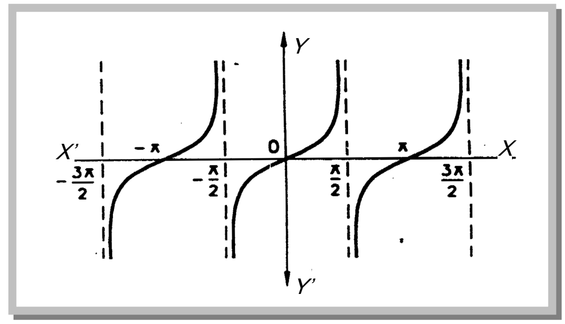


(b) **Cosine function:** The function that associates to each real number x to $\cos x$ is called the cosine function. Here x is the radian measure of the angle. The domain of the cosine function is R and the range is [– 1, 1].



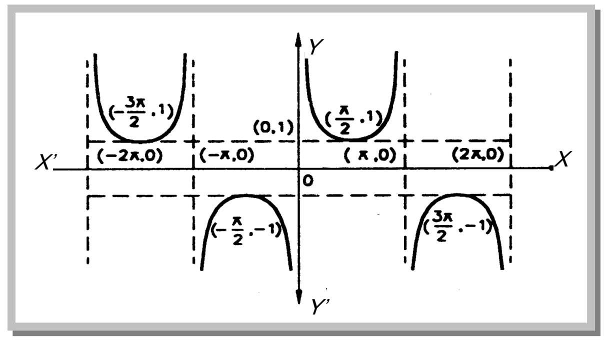
(c) **Tangent function:** The function that associates a real number x to tan x is called the tangent function.

Clearly, the tangent function is not defined at odd multiples of $\frac{\pi}{2}$ *i.e.,* $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ etc. So, the domain of the tangent function is $R - \{(2n+1)\frac{\pi}{2} | \ n \in I\}$. Since it takes every value between $-\infty$ and ∞. So, the range is *R*. Graph of $f(x) = \tan x$ is shown in figure.



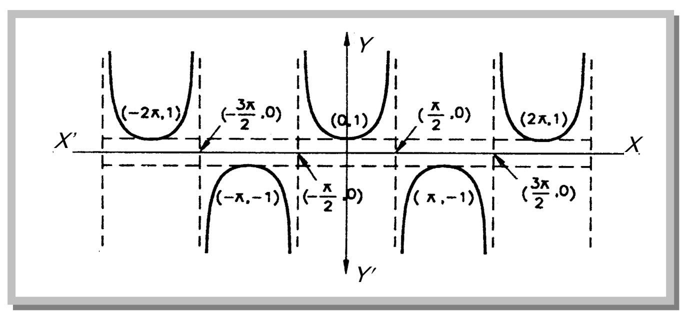
(d) **Cosecant function:** The function that associates a real number *x* to cosec*x* is called the cosecant function.

Clearly, cosec *x* is not defined at $x = n\pi, n \in I$. *i.e.,* $0, \pm\pi, \pm 2\pi, \pm 3\pi$ etc. So, its domain is $R - \{n\pi | \ n \in I\}$. Since $\operatorname{cosec} x \ge 1$ or $\operatorname{cosec} x \le -1$. Therefore, range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \operatorname{cosec} x$ is shown in figure.

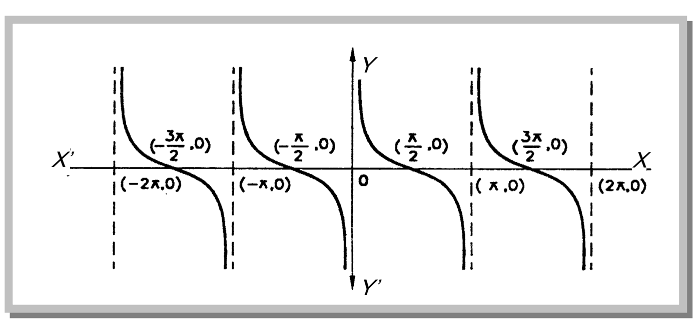


(e) **Secant function:** The function that associates a real number *x* to sec *x* is called the secant function.

Clearly, sec *x* is not defined at odd multiples of $\frac{\pi}{2}$ *i.e.,* $(2\pi+1)\frac{\pi}{2}$, where $n \in I$. so, its domain is $R - \{(2n+1)\frac{\pi}{2} | \ n \in I\}$. Also, $|\sec x| \ge 1$, therefore its range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \sec x$ is shown in figure.



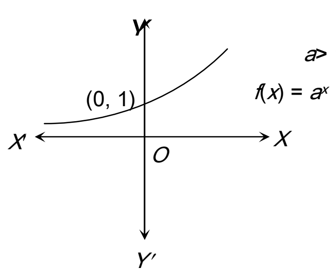
(f) **Cotangent function:** The function that associates a real number *x* to cot *x* is called the cotangent function. Clearly, cot *x* is not defined at $x = n\pi, n \in I$ i.e., at $n = 0, \pm\pi, \pm 2\pi$ etc. So, domain of $\cot x$ is $R - \{n\pi | \ n \in I\}$. the range of $f(x) = \cot x$ is *R* as is evident from its graph in figure.



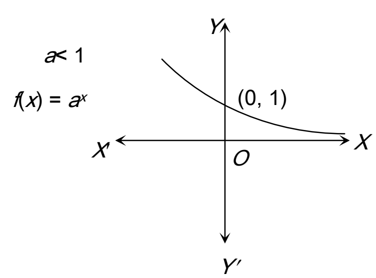
(ii) **Inverse trigonometric functions**

Function	**Domain**	**Range**	**Definition of the function**
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2,\ \pi/2]$	$y = \sin^{-1} x \Leftrightarrow x = \sin y$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$y = \cos^{-1} x \Leftrightarrow x = \cos y$
$\tan^{-1} x$	$(-\infty, \infty)$ or R	$(-\pi/2, \pi/2)$	$y = \tan^{-1} x \Leftrightarrow x = \tan y$
$\cot^{-1} x$	$(-\infty, \infty)$ or R	$(0, \pi)$	$y = \cot^{-1} x \Leftrightarrow x = \cot y$
$\text{cosec}^{-1} x$	$R - (-1, 1)$	$[-\pi/2,\ \pi/2] - \{0\}$	$y = \text{cosec}^{-1} x \Leftrightarrow x = \text{cosec} y$
$\sec^{-1} x$	$R - (-1, 1)$	$[0,\ \pi] - [\pi/2]$	$y = \sec^{-1} x \Leftrightarrow x = \sec y$

(iii) **Exponential function:** Let $a \neq 1$ be a positive real number. Then $f : R \to (0, \infty)$ defined by $f(x) = a^x$ is called exponential function. Its domain is R and range is $(0, \infty)$.

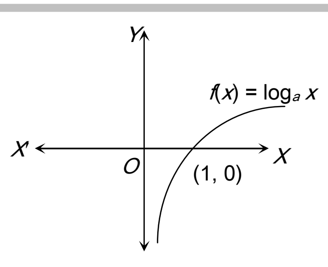


graph of $f(x) = a^x$, when $a > 1$

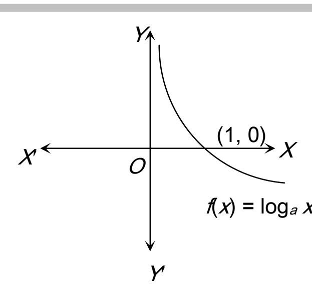


graph of $f(x) = a^x$, when $a < 1$

(iv) **Logarithmicfunction:**Let $a \neq 1$ be a positive real number. Then $f : (0, \infty) \to R$ defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is $(0, \infty)$and range is R.



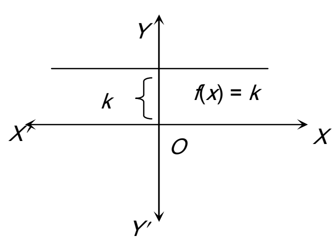
graph of $f(x) = \log_a x$, when $a > 1$



graph of $f(x) = \log_a x$, when $a < 1$

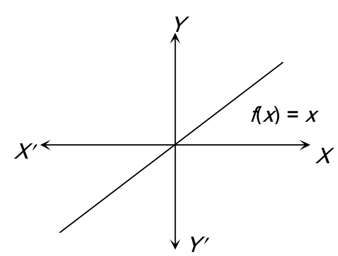
(8) **Explicit and implicit functions:** A function is said to be explicit if it can be expressed directly in terms of the independent variable. If the function cannot be expressed directly in terms of the independent variable or variables, then the function is said to be implicit. *E.g.* $y = \sin^{-1} x + \log x$ is explicit function, while $x^2 + y^2 = xy$ and $x^3 y^2 = (a - x)^2 (b - y)^2$ are implicit functions.

(9) **Constant function:** Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of f is the singleton set $\{k\}$. The graph of a constant function is a straight line parallel to x-axis as shown in figure and it is above or below the x-axis according as k is positive or negative. If $k = 0$, then the straight line coincides with x-axis.

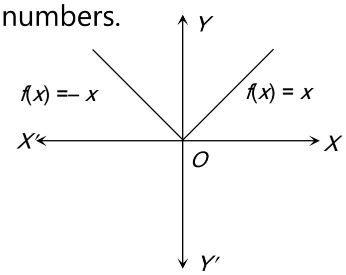


(10) **Identity function:** The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on R. Clearly, the domain and range of the identity function is R.

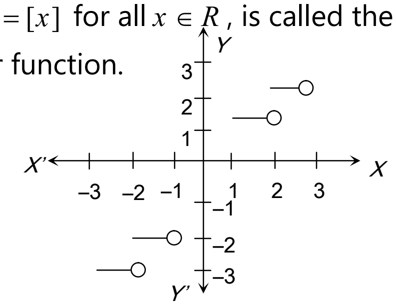
The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with positive direction of x-axis.



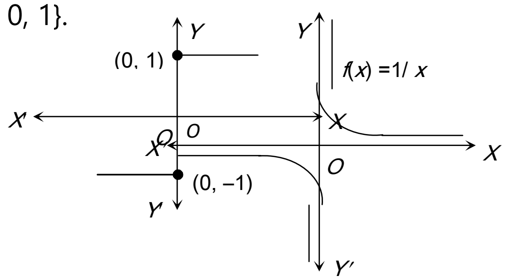
(11) **Modulus function:** The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \ge 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function. The domain of the modulus function is the set *R* of all real numbers and the range is the set of all non-negative real numbers.



(12) **Greatest integer function:** Let $f(x) = [x]$, where [*x*] denotes the greatest integer less than or equal to *x*. The domain is *R* and the range is *I*. *e.g.* [1.1] = 1, [2.2] = 2, [– 0.9] = –1, [– 2.1] = – 3 etc. The function *f* defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function.



(13) **Signum function:** The function defined by $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ or $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ is called the signum function. The domain is *R* and the range is the set {–1, 0, 1}.



(14) **Reciprocal function:** The function that associates each non-zero real number x to be reciprocal $\frac{1}{x}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to $R - \{0\}$ *i.e.,* the set of all non-zero real numbers. The graph is as shown.

Domain and Range of Some Standard Functions

Function	**Domain**	**Range**
Polynomial function	*R*	*R*
Identity function x	*R*	*R*
Constant function *K*	*R*	*{K}*
Reciprocal function $\frac{1}{x}$	R_0	R_0
$x^2, \lvert x \rvert$	*R*	$R^+ \cup \{0\}$

$x^3, x\|x\|$	R	R
Signum function	R	$\{-1, 0, 1\}$
$x+\|x\|$	R	$R^+ \cup \{0\}$
$x-\|x\|$	R	$R^- \cup \{0\}$
$[x]$	R	I
$x-[x]$	R	[0, 1)
$\sqrt{x}$	$[0, \infty)$	R
a^x	R	R^+
$\log x$	R^+	R
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R-\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...\right\}$	R
$\cot x$	$R-\{0, \pm\pi, \pm 2\pi, ...\}$	R
$\sec x$	$R-\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...\right\}$	$R-(-1, 1)$
$\operatorname{cosec} x$	$R-\{0, \pm\pi, \pm 2\pi, ...\}$	$R-(-1, 1)$
$\sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$R-(-1, 1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$

$\operatorname{cosec}^{-1}x$	*R*–$(-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Important Tips

- *Any function, which is entirely increasing or decreasing in the whole of a domain, is one-one.*
- *Any continuous function $f(x)$, which has at least one local maximum or local minimum, is many-one.*
- *If any line parallel to the x-axis cuts the graph of the function at most at one point, then the function is one-one and if there exists a line which is parallel to the x-axis and cuts the graph of the function in at least two points, then the function is many-one.*
- *Any polynomial function $f : R \rightarrow R$ is onto if degree of f is odd and into if degree of f is even.*
- *An into function can be made onto by redefining the co-domain as the range of the original function.*