The Mean Value Theorem (MVT)

Lagrange’s mean value theorem (MVT) states that if a function $f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b), then there is at least one point $x=c$ on this interval, such that
$f(b)-f(a)=f'(c)(b-a).$
This theorem (also known as First Mean Value Theorem) allows to express the increment of a function on an interval through the value of the derivative at an intermediate point of the segment.

Proof.
Consider the auxiliary function

$$F(x)=f(x)+\lambda x.$$

We choose a number λ such that the condition $F(a)=F(b)$ is satisfied. Then
$f(a)+\lambda a=f(b)+\lambda b,\Rightarrow f(b)-f(a)=\lambda(a-b),\Rightarrow \lambda=-f(b)-f(a)b-a.$
As a result, we have

$$F(x)=f(x)-f(b)-f(a)b-ax.$$

The function $F(x)$ is continuous on the closed interval $[a,b]$, differentiable on the open interval (a,b) and takes equal values at the endpoints of the interval. Therefore, it satisfies all the conditions of Rolle’s theorem. Then there is a point c in the interval (a,b) such that
$F'(c)=0.$
It follows that

$$f'(c)-f(b)-f(a)b-a=0$$

or

$$f(b)-f(a)=f'(c)(b-a).$$