Integrals with Infinite Limits (Improper Integral).

A definite integral $\int_a^b f(x)dx$ is called an improper integral, if

The range of integration is finite and the integrand is unbounded and/or the range of integration is infinite and the integrand is bounded.

e.g., The integral $\int_0^1 \frac{1}{x^2}dx$ is an improper integral, because the integrand is unbounded on [0, 1].

Infact, $\frac{1}{x^2} \to \infty$ as $x \to 0$. The integral $\int_0^\infty \frac{1}{1+x^2}dx$ is an improper integral, because the range of integration is not finite.

There are following two kinds of improper definite integrals:

(1) **Improper integral of first kind:**A definite integral $\int_a^b f(x)dx$ is called an improper integral of first kind if the range of integration is not finite (i.e., either $a \to \infty$ or $b \to \infty$ or $a \to \infty$ and $b \to \infty$) and the integrand f(x) is bounded on [a, b].

$\int_1^\infty \frac{1}{x^2}dx, \int_0^\infty \frac{1}{1+x^2}dx, \int_{-\infty}^\infty \frac{1}{1+x^2}dx, \int_1^\infty \frac{3x}{(1+2x)^3}dx$ are improper integrals of first kind.

Important Tips

- ☞ In an improper integral of first kind, the interval of integration is one of the following types [a, ∞), (–∞, b], (–∞, ∞).
- ☞ The improper integral $\int_a^\infty f(x)dx$ is said to be convergent, if $\lim_{k\to\infty}\int_a^k f(x)dx$ exists finitely and this limit is called the value of the improper integral. If $\lim_{k\to\infty}\int_a^k f(x)dx$ is either +∞ or –∞, then the integral is said to be divergent.
- ☞ The improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be convergent, if both the limits on the right-hand side exist finitely and are independent of each other. The improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be divergent if the right hand side is +∞ or –∞

(2) **Improper integral of second kind:** A definite integral $\int_a^b f(x)dx$ is called an improper integral of second kind if the range of integration [a, b] is finite and the integrand is unbounded at one or more points of [a, b].

If $\int_a^b f(x)dx$ is an improper integral of second kind, then a, b are finite real numbers and there exists at least one point $c \in [a,b]$ such that $f(x) \to +\infty$ or $f(x) \to -\infty$ as $x \to c$ i.e., f(x) has at least one point of finite discontinuity in [a, b].

For example:

(i) The integral $\int_1^3 \frac{1}{x-2}dx$, is an improper integral of second kind, because $\lim_{x\to 2}\left(\frac{1}{x-2}\right) = \infty$.

(ii) The integral $\int_0^1 \log x dx$; is an improper integral of second kind, because $\log x \to \infty \text{ as } x \to 0$.

(iii) The integral $\int_0^{2\pi} \frac{1}{1+\cos x}dx$, is an improper integral of second kind since integrand $\frac{1}{1+\cos x}$ becomes infinite at $x = \pi \in [0, 2\pi]$.

(iv) $\int_0^1 \frac{\sin x}{x}dx$, is a proper integral since $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Important Tips

- ☞ Let f(x) be bounded function defined on (a, b] such that a is the only point of infinite discontinuity of f(x) i.e., f(x) →∞ as x → a. Then the improper integral of f(x) on (a, b] is denoted by $\int_a^b f(x)dx$ and is defined as $\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^b f(x)dx$. Provided that the limit on right hand side exists. If l denotes the limit on right hand side, then the improper integral $\int_a^b f(x)dx$ is said to converge to l, when l is finite. If l = + ∞ or l = –∞, then the integral is said to be a divergent integral.
- ☞ Let f(x) be bounded function defined on [a, b) such that b is the only point of infinite discontinuity of f(x) i.e. f(x) →∞ as x → b. Then the improper integral of f(x) on [a, b) is denoted by $\int_a^b f(x)dx$ and is defined as $\int_a^b f(x)dx = \lim_{\varepsilon \to 0}\int_a^{b-\varepsilon} f(x)dx$
Provided that the limit on right hand side exists finitely. If l denotes the limit on right hand side, then the improper integral $\int_a^b f(x)dx$ is said to converge to l, when l is finite.

If $l = +\infty$ or $l = -\infty$, then the integral is said to be a divergent integral.

☞ Let f(x) be a bounded function defined on (a, b) such that a and b are only two points of infinite discontinuity of f(x) i.e., f(a) →∞, f(b) →∞.

Then the improper integral of f(x) on (a, b) is denoted by $\int_a^b f(x)dx$ and is defined as

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{c} f(x)dx + \lim_{\delta \to 0} \int_a^{b-\delta} f(x)dx, a < c < b$$

Provided that both the limits on right hand side exist.

☞ Let f(x) be a bounded function defined [a, b]-{c}, c∈[a, b] and c is the only point of infinite discontinuity of f(x) i.e. f(c)→∞. Then the improper integral of f(x) on [a, b] – {c} is denoted by $\int_a^b f(x)dx$ and is defined as $\int_a^b f(x)dx = \lim_{x \to 0} \int_a^{c-x} f(x)dx + \lim_{\delta \to 0} \int_{c+\delta}^{b} f(x)dx$

Provided that both the limits on right hand side exist finitely. The improper integral $\int_a^b f(x)dx$ is said to be convergent if both the limits on the right hand side exist finitely.

☞ If either of the two or both the limits on RHS are ±∞, then the integral is said to be divergent.