Definite Integral as the Limit of a Sum.

Let f(x) be a single valued continuous function defined in the interval $a \le x \le b,$ where a and b are both finite. Let this interval be divided into n equal sub-intervals, each of width h by inserting (n – 1) points $a+h, a+2h, a+3h...... a+(n-1)h$ between a and b. Then $nh = b-a$.

Now, we form the sum $hf(a)+hf(a+h)+hf(a+2h)+........+hf(a+rh)+......+hf[a+(n-1)h]$

$= h[f(a)+f(a+h)+f(a+2h)+.....+f(a+rh)+....+f\{a+(n-1)h\}]$

$= h\sum_{r=0}^{n-1} f(a+rh)$

Where, $a+nh=b \Rightarrow nh=b-a$

The $\lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh),$ if it exists is called the **definite integral** of f(x) with respect to x between the limits a and b and we denote it by the symbol $\int_a^b f(x)dx$.

Thus, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+......+f\{a+(n-1)h\}] \Rightarrow \int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$

Where, $nh = b-a,$ a and b being the limits of integration.

The process of evaluating a definite integral by using the above definition is called integration from the first principle or integration as the limit of a sum.

Important Tips

☞ In finding the above sum, we have taken the left end points of the subintervals. We can take the right end points of the sub-intervals throughout.

Then we have, $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a+h)+f(a+2h)+.....+f(a+nh)]$, $\Rightarrow \int_a^b f(x)dx = h\sum_{r=1}^{n} f(a+rh)$

Where, nh = b – a.

☞ $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}(\beta>\alpha)=\pi$

☞ $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\, dx = \frac{\pi}{8}(\beta-\alpha)^2$

☞ $\int_a^b \sqrt{\frac{x-a}{b-x}}\, dx = \frac{\pi}{2}(b-a)$

☞ $\int_a^b f(x)\, dx = \frac{1}{n}\int_{na}^{nb} f(x)\, dx$

☞ $\int_{a-c}^{b-c} f(x+c)\, dx = \int_a^b f(x)\, dx$ or $\int_{a+c}^{b+c} f(x-c)\, dx = \int_a^b f(x)\, dx$

Some useful results for evaluation of definite integrals as limit for sums

(i) $1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(iv) $a + ar + ar^2 + \ldots\ldots + ar^{n-1} = \frac{a(r^n - 1)}{r-1}, r \neq 1,\ r > 1$

(v) $a + ar + ar^2 + \ldots\ldots\ \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r},\ r \neq 1,\ r < 1$

(vi) $\sin a + \sin(a+h) + \ldots\ldots + \sin[a+(n-1)h] = \sum_{r=0}^{n-1}[\sin(a+nh)] = \frac{\sin\left\{a + \left(\frac{n-1}{2}\right)h\right\}\sin\left\{\frac{nh}{2}\right\}}{\sin\left(\frac{h}{2}\right)}$

(vii) $\cos a + \cos(a+h) + \cos(a+2h) + \ldots\ldots + \cos[a+(n-1)h]$

$$= \sum_{r=0}^{n-1}[\cos(a+nh)] = \frac{\cos\left\{a + \left(\frac{n-1}{2}\right)h\right\}\sin\left\{\frac{nh}{2}\right\}}{\sin\left(\frac{h}{2}\right)}$$

(viii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots\ldots\ \ldots\ldots\ \infty = \frac{\pi^2}{12}$

(ix) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \ldots\ldots\ \ldots\ \infty = \frac{\pi^2}{6}$

(x) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ \ldots\ldots\ \infty = \frac{\pi^2}{8}$

(xi) $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty=\frac{\pi^2}{24}$

(xii) $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2}$

(xiii) $\cosh\theta=\frac{e^{\theta}+e^{-\theta}}{2}$ and $\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}$