Properties of Definite Integral.

(1) $\int_a^b f(x)dx = \int_a^b f(t)dt$ i.e., The value of a definite integral remains unchanged if its variable is replaced by any other symbol.

(2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ i.e., by the interchange in the limits of definite integral, the sign of the integral is changed.

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, (where a < c < b)

or $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_n}^b f(x)dx$; (where $a < c_1 < c_2 < .... c_n < b$)

Generally this property is used when the integrand has two or more rules in the integration interval.

Important Tips

☞ $\int_a^b (|x-a|+|x-b|)\ dx = (b-a)^2$

Note: Property (3) is useful when $f(x)$ is not continuous in [a, b] because we can break up the integral into several integrals at the points of discontinuity so that the function is continuous in the sub-intervals. The expression for $f(x)$ changes at the end points of each of the sub-interval. You might at times be puzzled about the end points as limits of integration. You may not be sure for x = 0 as the limit of the first integral or the next one. In fact, it is immaterial, as the area of the line is always zero. Therefore, even if you write $\int_{-1}^0 (1-2x)\,dx$, whereas 0 is not included in its domain it is deemed to be understood that you are approaching x = 0 from the left in the first integral and from right in the second integral. Similarly for x = 1.

(4) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$: This property can be used only when lower limit is zero. It is generally used for those complicated integrals whose denominators are unchanged when x is replaced by (a – x).

Following integrals can be obtained with the help of above property.

(i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} dx = \frac{\pi}{4}$

(ii) $\int_0^{\pi/2} \frac{\tan^n x}{1 + \tan^n x} dx = \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} dx = \frac{\pi}{4}$

(iii) $\int_0^{\pi/2} \frac{1}{1 + \tan^n x} dx = \int_0^{\pi/2} \frac{1}{1 + \cot^n x} dx = \frac{\pi}{4}$

(iv) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \mathrm{cosec}^n x} dx = \int_0^{\pi/2} \frac{\mathrm{cosec}^n x}{\mathrm{cosec}^n x + \sec^n x} dx = \frac{\pi}{4}$

(v) $\int_0^{\pi/2} f(\sin 2x) \sin x dx = \int_0^{\pi/2} f(\sin 2x) \cos x dx$

(vi) $\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx$

(vii) $\int_0^{\pi/2} f(\tan x)dx = \int_0^{\pi/2} f(\cot x)dx$

(viii) $\int_0^1 f(\log x)dx = \int_0^1 f[\log(1-x)]dx$

(ix) $\int_0^{\pi/2} \log \tan x dx = \int_0^{\pi/2} \log \cot x dx$

(x) $\int_0^{\pi/4} \log(1 + \tan x)dx = \frac{\pi}{8} \log 2$

(xi) $\int_0^{\pi/2} \log\sin x dx = \int_0^{\pi/2} \log\cos x dx = \frac{-\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$

(xii) $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{a\sec x + b\,\mathrm{cosec}\, x}{\sec x + \mathrm{cosec}\, x} dx = \int_0^{\pi/2} \frac{a\tan x + b\cot x}{\tan x + \cot x} dx = \frac{\pi}{4}(a+b)$

(5) $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x) + f(-x)\,dx$.

In special case: $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even function or } f(-x) = f(x) \\ 0 \quad ,\text{if } f(x) \text{ is odd is odd function or } f(-x) = -f(x) \end{cases}$

This property is generally used when integrand is either even or odd function of x.

(6) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)\,dx$

In particular, $\int_0^{2a} f(x)dx = \begin{cases} 0 \quad , \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \end{cases}$

It is generally used to make half the upper limit.

(7) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Note: $\int_a^b \frac{f(x)dx}{f(x)+f(a+b-x)} = \frac{1}{2}(b-a)$

(8) $\int_0^a x\, f(x)dx = \frac{1}{2}a\int_0^a f(x)dx$ if $f(a-x) = f(x)$

(9) If $f(x)$ is a periodic function with period T, then $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$,

Deduction: If $f(x)$ is a periodic function with period T and $a \in R^{+}$, then $\int_{nT}^{a+nT} f(x)dx = \int_{0}^{a} f(x)dx$

(10)

(i) If $f(x)$ is a periodic function with period T, then

$\int_{a}^{a+nT} f(x)\,dx = n\int_{0}^{T} f(x)\,dx$ where $n \in I$

(a) In particular, if a = 0

$\int_{0}^{nT} f(x)\,dx = n\int_{0}^{T} f(x)\,dx,$ where $n \in I$

(b) If n = 1, $\int_{0}^{a+T} f(x)\,dx = \int_{0}^{T} f(x)\,dx,$

(i) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_{0}^{T} f(x)\,dx,$ where n, $m \in I$

(ii) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_{a}^{b} f(x)\,dx,$ where $n \,\varepsilon\, I$

(11) If $f(x)$ is a periodic function with period T, then $\int_{a}^{a+T} f(x)$ is independent of a.

(12) $\int_{a}^{b} f(x)\,dx = (b-a)\int_{0}^{1} f\big((b-a)x+a\big)\,dx$

(13) If $f(t)$ is an odd function, then $\phi(x) = \int_{a}^{x} f(t)\,dt$ is an even function

(14) If $f(x)$ is an even function, then $\phi(x) = \int_{0}^{x} f(t)\,dt$ is on odd function.

Note: If f(t) is an even function, then for a non zero 'a', $\int_0^x f(t)\,dt$ is not necessarily an odd function. It will be odd function if $\int_0^a f(t)\,dt = 0$

Important Tips

- Every continuous function defined on [a, b] is integrable over [a, b].
- Every monotonic function defined on [a, b] is integrable over [a, b].
- If f(x) is a continuous function defined on [a, b], then there exists $c \in (a,b)$ such that $\int_a^b f(x)dx = f(c).(b-a)$.

  The number $f(c) = \frac{1}{(b-a)}\int_a^b f(x)dx$ is called the mean value of the function $f(x)$ on the interval [a, b].
- If f is continuous on [a, b], then the integral function g defined by g(x) = $\int_a^x f(t)dt$ for $x \in [a,b]$ is derivable on [a, b] and $g'(x) = f(x)$ for all $x \in [a,b]$.
- If m and M are the smallest and greatest values of a function f(x) on an interval [a, b], then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$
- If the function $\varphi(x)$ and $\psi(x)$, are defined on [a, b] and differentiable at a point $x\varepsilon(a,b)$ and f(t) is continuous for $\phi(a) \le t \le \psi(b)$, then $\left(\int_{\varphi(x)}^{\psi(x)} f(t)dt\right) = f(\psi(x))\psi'(x) - f(\varphi(x))\varphi'(x)$

  $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$
- If $f^2(x)$ and $g^2(x)$ are integrable on [a, b], then $\left|\int_a^b f(x)g(x)dx\right| \le \left(\int_a^b f^2(x)dx\right)^{1/2}\left(\int_a^b g^2(x)dx\right)^{1/2}$
- **Change of variables**: If the function f(x) is continuous on [a, b] and the function $x = \varphi(t)$ is continuously differentiable on the interval $[t_1, t_2]$ and $a = \varphi(t_1), b = \varphi(t_2)$, then $\int_a^b f(x)dx = \int_{t_1}^{t_2} f(\varphi(t))\varphi'(t)dt$.
- Let a function $f(x,\alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then for any $\alpha \in [c,d]$, if $I(\alpha) = \int_a^b f(x,\alpha)dx$, then $I'(\alpha) = \int_a^b f'(x,\alpha)dx$,

  Where $I'(\alpha)$ is the derivative of $I(\alpha)$ w.r.t. α and $f'(x,\alpha)$ is the derivative of $f(x,\alpha)$ w.r.t. α, keeping x constant.
- For a given function $f(x)$ continuous on [a, b] if you are able to find two continuous function $f_1(x)$ and $f_2(x)$ on [a, b] such that $f_1(x) \le f(x) \le f_2(x)\ \forall x \in [a,b]$, then

$$\int_a^b f_1(x)dx \le \int_a^b f(x)dx \le \int_a^b f_2(x)dx$$