Mathematical Induction:

History of Mathematical Induction

Earlier, the mathematical induction was found in the Euclid’s proof of the infinite prime numbers.

Another proof of mathematical induction was there in the al-Fakhri written by al-Karaji which was used by him to prove the properties of Pascal's triangle.

What is Induction in math?

Mathematical induction is a special method or technique to prove the statements given in consideration. Generally, it is used to prove the statements of the set of natural numbers.

As in dominoes if we knock the first domino then it shows that the first domino falls, and if we knock any one domino then the next to it will also fall.

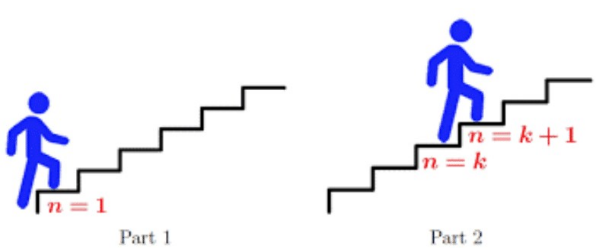
Principle of Mathematical Induction

Mathematical induction is used to prove that the given statement is true or not. It uses 2 steps to prove it.

First Principle of Mathematical Induction

- **Base Case**: The given statement is correct for first natural number that is, for n=1, p (1) is true.
- **Inductive Step**: If the given statement is true for any natural number like n = k then it will be correct for n = k + 1 also that is, if p (k) is true then p (k+1) will also be true.

The first principle of mathematical induction says that if both the above steps are proven then p (n) is true for all natural numbers.



Second Principle of Mathematical Induction

It is more powerful than the first principle. It is sometimes called as strong induction. It doesn't need a base case in special case otherwise it may require to show with more than one base case. In the recursive step to prove that k +1 is true, first we need to prove that the statement is correct for all the numbers less than k+1 also.

First Principle of Mathematical Induction

- Steps of Mathematical Induction
- Examples
- What is Induction Hypothesis?
- Properties of Principle of Mathematical Induction

Steps of Mathematical Induction

The mathematical induction can be proved in four steps:

- **Given**: the statement you want to prove.
- **Beginning Step**: Show p (1) is true. Let n =1 and solve.
- **Assumption Step (Induction Hypothesis)**: Assume p (k) is true. Let n = k and solve.
- **Induction step**: Show if p (k) is true then p (k+1) is also true. Let n = k+1 and solve.

Write the proof that p (n) is true.

Examples

<u>**Example 1**</u>

Prove that for any natural number n, the sum of n natural numbers is

$$\frac{n\,(n+1)}{2}$$

Solution: Given the statement which we need to prove

$$1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}$$

Step 1: Base Step

Show the statement holds for n =1

1 = 1(1+1)/2 =2/2 = 1

As the LHS = RHS

This shows that p (1) is true.

Step 2: Inductive Step

Show if p (k) is right then p (k+1) is also right.

Assume p (k) is right

Let n = k

$$P\,(k) = 1 + 2 + 3 + \ldots + k = \frac{k\,(k + 1)}{2}$$

Now we have to prove that it is true for the next number also i.e. k+1

Let n = k+1

$$P\,(k + 1) = 1 + 2 + 3 + \ldots + k + 1 = \frac{(k + 1)\,[(k + 1) + 1]}{2}$$

Using the induction hypothesis which assumes that p (k) is true, we can rewrite it as:

$$1 + 2 + 3 + \ldots + k + k+1 = \frac{k(k+1)}{2} + k + 1 \text{ (by adding k+1 to the both sides)}$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2} \text{ (taking k+1 as common)}$$
$$= \frac{(k+1)[(k+1)+1]}{2}$$

This shows that p (k+1) is true.

The principle of mathematical induction shows that if both the above steps are true then p (n) is true for all n natural numbers.

<u>Example 2</u>

Prove using mathematical induction that

$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$,for all positive integers n.

Solution: Now we will use base step and induction step to prove it.

Step 1

Let n = 1

$$\frac{1(1+1)(2*1+1)}{6} = \frac{1*2*3}{6} = \frac{6}{6} = 1$$

The formula is true for n = 1 that is, the statement is true for p (1)

Step 2

Assume p (k) and prove for p (k+1).

Use P(k) to show that P(k + 1) is true.

$$\text{Assume: } 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

$$\text{Prove: } 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$$
$$= \frac{(k+1)(k+2)\left[2(k+1)+1\right]}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}.$$

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 & \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6} \\ &= \frac{(k+1)\left(2k^2 + 7k + 6\right)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

Therefore, by the Principle of Mathematical Induction,

$1^2 + 2^2 + 3^2 + \cdots + n^2 = n\,(n + 1)\,(2n + 1) / 6$, is true for all positive integers n.

<u>Example 3</u>

$$P(n) : \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Solution:

Step 1

Show p (1) is true.

$$P(1) : \left(\frac{a}{b}\right)^1 = \frac{a}{b} = \frac{a^1}{b^1}$$

So p (1) is true.

Step 2

Assume p (k) is true.

$$P(k) : \left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$$

Show if p (k) is true then p (k+1) is also true.

Now use p (k) and p (1) to prove it.

$$P(k+1) : \left(\frac{a}{b}\right)^{k+1} = \left(\frac{a}{b}\right)^k \left(\frac{a}{b}\right)^1 = \frac{a^k}{b^k} \cdot \frac{a^1}{b^1}$$

$$= \underset{P(k)}{\uparrow} \quad \underset{P(1)}{\uparrow} \quad \underset{P(k)}{\uparrow} \quad \underset{P(1)}{\uparrow}$$

$$= \frac{a^k a^1}{b^k b^1} = \frac{a^{k+1}}{b^{k+1}}$$

This shows that this statement is true for p (k+1) also.

Hence p (n) is true.

What is Induction Hypothesis?

In the principle of mathematical induction when we reach to the inductive step, we need to assume p (k) to prove the statement for p (k+1).that assumption is induction hypothesis.

__Example__

Prove using induction hypothesis: sum of odd natural numbers is n^2.

Solution: **Theorem**: $1 + 3 + 5 + 7 + \ldots + 2n - 1 = n^2$

Step 1: Base step

Show the statement holds for n =1

$1 = n^2 = 1$

As the LHS = RHS

Properties of Principle of Mathematical Induction

- **Base Step** is basically a statement of fact which shows that the statement is true for the first number of the set of natural numbers. If it is given that the statement is true for $n \geq 3$, then we will start with $n = 3$ and verify that for $n = 3$, p (3) is true.
- **Inductive Step** is the conditional property of mathematical induction. As it is not the statement which state that $n = k$, but it is the condition that if statement is true for $n = k$ then it will be true for $n = k + 1$ also.

Second Principle of Mathematical Induction

If we want to prove that every positive integer can be factored into prime.

Let M is a subset of the set of all natural numbers.

$1, 2, \ldots, l \in M$ and $l \in N$, and

for any $k \geq l$, we have $1, 2, \ldots, k \in M$ implies $k + 1 \in M$.

Then $M = N$.

Mathematical Induction

	First Principle of Mathematical Induction	Second Principle of Mathematical Induction
1.	It is also known as weak induction or ordinary induction.	It is also known as strong induction or complete induction.
2.	It is difficult form of induction.	It is the easiest form of induction.
3.	In base case, one has to show for p(1)	In base case ,one has to show for p(0) and sometimes some extra base case also like p(1) etc.
4.	In inductive step, it assumes p (n).	In recursive step, one proves the statement P (n + 1), under the assumption that P (n) is true for all natural numbers less than n + 1.