Escape Velocity.

The minimum velocity with which a body must be projected up so as to enable it to just overcome the gravitational pull, is known as escape velocity.

The work done to displace a body from the surface of earth (r = R) to infinity ($r = \infty$) is

$$W = \int_R^\infty \frac{GMm}{x^2} dx = -GMm\left[\frac{1}{\infty} - \frac{1}{R}\right]$$

$\Rightarrow$ $$W = \frac{GMm}{R}$$

This work required to project the body so as to escape the gravitational pull is performed on the body by providing an equal amount of kinetic energy to it at the surface of the earth.

If v_e is the required escape velocity, then kinetic energy which should be given to the body is $\frac{1}{2}mv_e^2$

$\therefore$ $$\frac{1}{2}mv_e^2 = \frac{GMm}{R} \Rightarrow v_e = \sqrt{\frac{2GM}{R}}$$

$$\Rightarrow v_e = \sqrt{2gR}$$ [As $GM = gR^2$]

or $$v_e = \sqrt{2 \times \frac{4}{3}\pi\rho GR \times R} \Rightarrow v_e = R\sqrt{\frac{8}{3}\pi G\rho}$$ [As $g = \frac{4}{3}\pi\rho GR$]

Important points

(i) Escape velocity is independent of the mass and direction of projection of the body.

(ii) Escape velocity depends on the reference body. Greater the value of (M/R) or (gR) for a planet, greater will be escape velocity.

(iii) For the earth as $g = 9.8 m/s^2$ and $R = 6400\ km$

$$\therefore v_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = 11.2 km/\sec$$

(iv) A planet will have atmosphere if the velocity of molecule in its atmosphere $\left[v_{rms} = \sqrt{\frac{3RT}{M}}\right]$ is lesser than escape velocity. This is why earth has atmosphere (as at earth $v_{rms} < v_e$) while moon has no atmosphere (as at moon $v_{rms} < v_e$)

(v) If body projected with velocity lesser than escape velocity ($v < v_e$) it will reach a certain maximum height and then may either move in an orbit around the planet or may fall down back to the planet.

(vi) Maximum height attained by body: Let a projection velocity of body (mass m) is v, so that is attains a maximum height h. At maximum height, the velocity of particle is zero, so kinetic energy is zero.

By the law of conservation of energy

Total energy at surface = Total energy at height h.

$$\Rightarrow \quad -\frac{GMm}{R} + \frac{1}{2}mv^2 = -\frac{GMm}{R+h} + 0$$

$$\Rightarrow \quad \frac{v^2}{2} = GM\left[\frac{1}{R} - \frac{1}{R+h}\right] = \frac{GMh}{R(R+h)}$$

$$\Rightarrow \quad \frac{2GM}{v^2R} = \frac{R+h}{h} = 1 + \frac{R}{h}$$

$$\Rightarrow \quad h = \frac{R}{\left(\frac{2GM}{v^2R} - 1\right)} = \frac{R}{\frac{v_e^2}{v^2} - 1} = R\left[\frac{v^2}{v_e^2 - v^2}\right]$$

[As $v_e = \sqrt{\frac{2GM}{R}} \therefore \frac{2GM}{R} = v_e^2$]

(vii) If a body is project with velocity greater than escape velocity $(v > v_e)$ then by conservation of energy.

Total energy at surface = Total energy at infinite

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}m(v')^2 + 0$$

i.e., $(v')^2 = v^2 - \frac{2GM}{R} \Rightarrow v'^2 = v^2 - v_e^2$ [As $\frac{2GM}{R} = v_e^2$]

$\therefore \quad v' = \sqrt{v^2 - v_e^2}$

i.e., the body will move in interplanetary or inter stellar space with velocity $\sqrt{v^2 - v_e^2}$.

(viii) Energy to be given to a stationary object on the surface of earth so that its total energy becomes zero, is called escape energy.

Total energy at the surface of the earth $= KE + PE = 0 - \frac{GMm}{R}$

$\therefore$ Escape energy $= \frac{GMm}{R}$

(ix) If the escape velocity of a body is equal to the velocity of light then from such bodies nothing can escape, not even light. Such bodies are called black holes.

The radius of a black hole is given as

$R = \frac{2GM}{C^2}$ [As $C = \sqrt{\frac{2GM}{R}}$, where C is the velocity of light]