

## Hydrostatics

It is common knowledge that the pressure of the atmosphere (about  $10^5$  newtons per square metre) is due to the weight of air above the Earth's surface, that this pressure falls as one climbs upward, and, correspondingly, that pressure increases as one dives deeper into a lake (or comparable body of water). Mathematically, the rate at which the pressure in a stationary fluid varies with height  $z$  in a vertical gravitational field of strength  $g$  is given by

$$\frac{dp}{dz} = -\rho g. \quad (123)$$

If  $\rho$  and  $g$  are both independent of  $z$ , as is more or less the case in lakes, then

$$p(z) = p(0) - \rho g z. \quad (124)$$

This means that, since  $\rho$  is about  $10^3$  kilograms per cubic metre for water and  $g$  is about 10 metres per second squared, the pressure is already twice the atmospheric value at a depth of 10 metres. Applied to the atmosphere, equation (124) would imply that the pressure falls to zero at a height of about 10 kilometres. In the atmosphere, however, the variation of  $\rho$  with  $z$  is far from negligible and (124) is unreliable as a consequence; a better approximation is given below in the section Hydrodynamics: Compressible flow in gases.

$$p(z) = p(0) - \rho g z. \quad (124)$$

## Differential manometers

Instruments for comparing pressures are called differential manometers, and the simplest such instrument is a U-tube containing liquid, as shown in Figure 1A. The two pressures of interest,  $p_1$  and  $p_2$ , are transmitted to the two ends of the liquid column through an inert gas—the density of which is negligible by comparison with the liquid density,  $\rho$ —and the difference of height,  $h$ , of the two menisci is measured. It is a consequence of (124) that

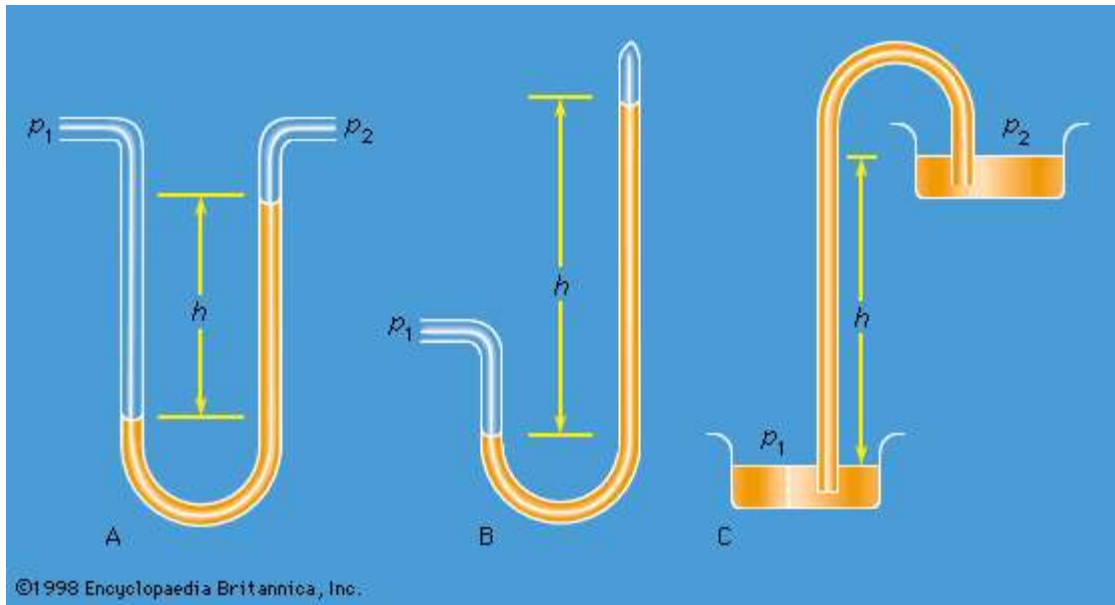


Figure 1: Schematic representations of (A) a differential manometer, (B) a Torricellian barometer, and (C) a siphon. *Encyclopaedia Britannica, Inc.*

$$p_1 - p_2 = \rho gh. \quad (125)$$

A barometer for measuring the pressure of the atmosphere in absolute terms is simply a manometer in which  $p_2$  is made zero, or as close to zero as is feasible. The barometer invented in the 17th century by the Italian physicist and mathematician Evangelista Torricelli, and still in use today, is a U-tube that is sealed at one end (see Figure 1B). It may be filled with liquid, with the sealed end downward, and then inverted. On inversion, a negative pressure may momentarily develop at the top of the liquid column if the column is long enough; however, cavitation normally occurs there and the column falls away from the sealed end of the tube, as shown in the figure. Between the two exists what Torricelli thought of as a vacuum, though it may be very far from that condition if the barometer has been filled without scrupulous precautions to ensure that all dissolved or adsorbed gases, which would otherwise collect in this space, have first been removed. Even if no contaminating gas is present, the Torricellian vacuum always contains the vapour of the liquid, and this exerts a pressure which may be small but is never quite zero. The liquid conventionally used in a Torricelli barometer is of course mercury, which has a low vapour pressure and a high density. The high density means that  $h$  is only about 760 millimetres; if water were used, it would have to be about 10 metres instead.



**Torricelli, Evangelista** Italian physicist and mathematician Evangelista Torricelli, inventor of the mercury barometer. *Photos.com/Thinkstock*

Figure 1C illustrates the principle of the siphon. The top container is open to the atmosphere, and the pressure in it,  $p_2$ , is therefore atmospheric. To balance this and the weight of the liquid column in between, the pressure  $p_1$  in the bottom container ought to be greater by  $\rho gh$ . If the bottom container is also open to the atmosphere, then equilibrium is clearly impossible; the weight of the liquid column prevails and causes the liquid to flow downward. The siphon operates only as long as the column is continuous; it fails if a bubble of gas collects in the tube or if cavitation occurs. Cavitation therefore limits the level differences over which siphons can be used, and it also limits (to about 10 metres) the depth of wells from which water can be pumped using suction alone.

### **Archimedes' principle**

Consider now a cube of side  $d$  totally immersed in liquid with its top and bottom faces horizontal. The pressure on the bottom face will be higher than on the top by  $\rho gd$ , and, since pressure is force per unit area and the area of a cube face is  $d^2$ , the resultant

upthrust on the cube is  $\rho g d^3$ . This is a simple example of the so-called Archimedes' principle, which states that the upthrust experienced by a submerged or floating body is always equal to the weight of the liquid that the body displaces.

As Archimedes must have realized, there is no need to prove this by detailed examination of the pressure difference between top and bottom. It is obviously true, whatever the body's shape. It is obvious because, if the solid body could somehow be removed and if the cavity thereby created could somehow be filled with more fluid instead, the whole system would still be in equilibrium. The extra fluid would, however, then be experiencing the upthrust previously experienced by the solid body, and it would not be in equilibrium unless this were just sufficient to balance its weight.

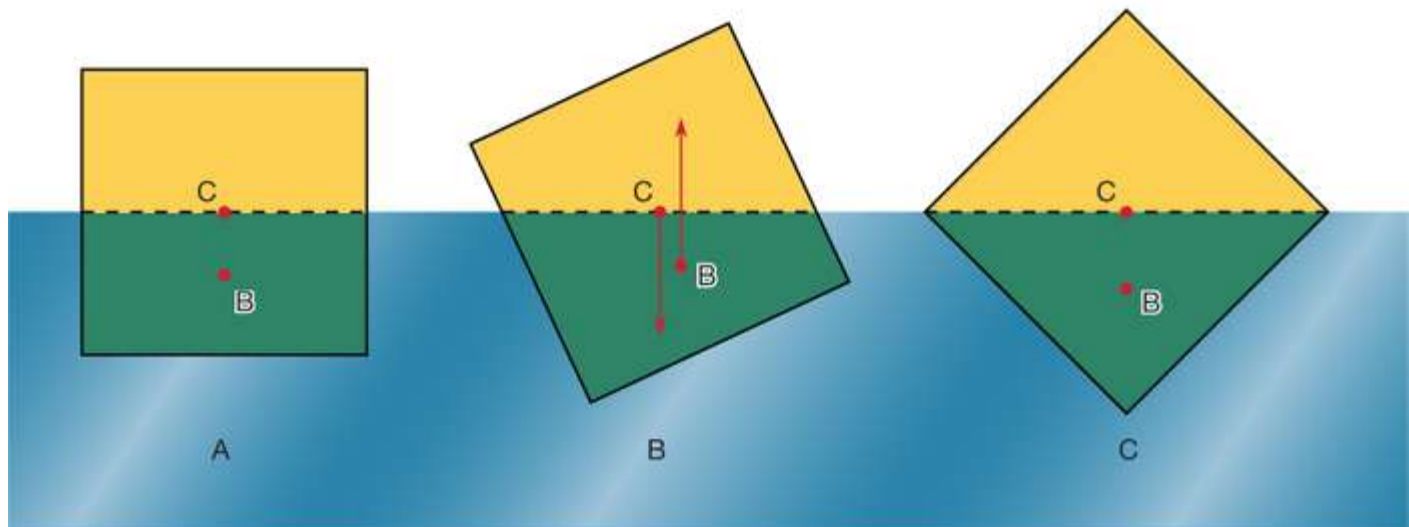
Archimedes' problem was to discover, by what would nowadays be called a nondestructive test, whether the crown of King Hieron II was made of pure gold or of gold diluted with silver. He understood that the pure metal and the alloy would differ in density and that he could determine the density of the crown by weighing it to find its mass and making a separate measurement of its volume. Perhaps the inspiration that struck him (in his bath) was that one can find the volume of any object by submerging it in liquid in something like a measuring cylinder (*i.e.*, in a container with vertical sides that have been suitably graduated) and measuring the displacement of the liquid surface. If so, he no doubt realized soon afterward that a more elegant and more accurate method for determining density can be based on the principle that bears his name. This method involves weighing the object twice, first, when it is suspended in a vacuum (suspension in air will normally suffice) and, second, when it is totally submerged in a liquid of density  $\rho$ . If the density of the object is  $\rho'$ , the ratio between the two weights must be

$$\frac{W_2}{W_1} = \frac{(\rho' - \rho)}{\rho'}. \quad (126)$$

If  $\rho'$  is less than  $\rho$ , then  $W_2$ , according to equation (126), is negative. What that means is that the object does not submerge of its own accord; it has to be pushed downward to make it do so. If an object with a mean density less than that of water is placed in a lake and not subjected to any downward force other than its own weight, it naturally floats on the surface, and Archimedes' principle shows that in equilibrium the volume of water which it displaces is a fraction  $\rho'/\rho$  of its own volume. A hydrometer is an object graduated in such a way that this fraction may be measured. By floating a hydrometer first in water of density  $\rho_0$  and then in some other liquid of density  $\rho_1$  and comparing the readings, one may determine the ratio  $\rho_1/\rho_0$ —*i.e.*, the specific gravity of the other liquid.

$$\frac{W_2}{W_1} = \frac{(\rho' - \rho)}{\rho'}. \quad (126)$$

In what orientation an object floats is a matter of grave concern to those who design boats and those who travel in them. A simple example will suffice to illustrate the factors that determine orientation. Figure 2 shows three of the many possible orientations that a uniform square prism might adopt when floating, with half its volume submerged in a liquid for which  $\rho = 2\rho'$ ; they are separated by rotations of  $22.5^\circ$ . In each of these diagrams, C is the centre of mass of the prism, and B, a point known as the centre of buoyancy, is the centre of mass of the displaced water. The distributed forces acting on the prism are equivalent to its weight acting downward through C and to the equal weight of the displaced water acting upward through B. In general, therefore, the prism experiences a torque. In Figure 2B the torque is counterclockwise, and so it turns the prism away from 2A and toward 2C. In 2C the torque vanishes because B is now vertically below C, and this is the orientation that corresponds to stable equilibrium. The torque also vanishes in 2A, and the prism can in principle remain indefinitely in that orientation as well; the equilibrium in this case, however, is unstable, and the slightest disturbance will cause the prism to topple one way or the other. In fact, the potential energy of the system, which increases in a linear fashion with the difference in height between C and B, is at its smallest in orientation 2C and at its largest in orientation 2A. To improve the stability of a floating object one should, if possible, lower C relative to B. In the case of a boat, this may be done by redistributing the load inside.

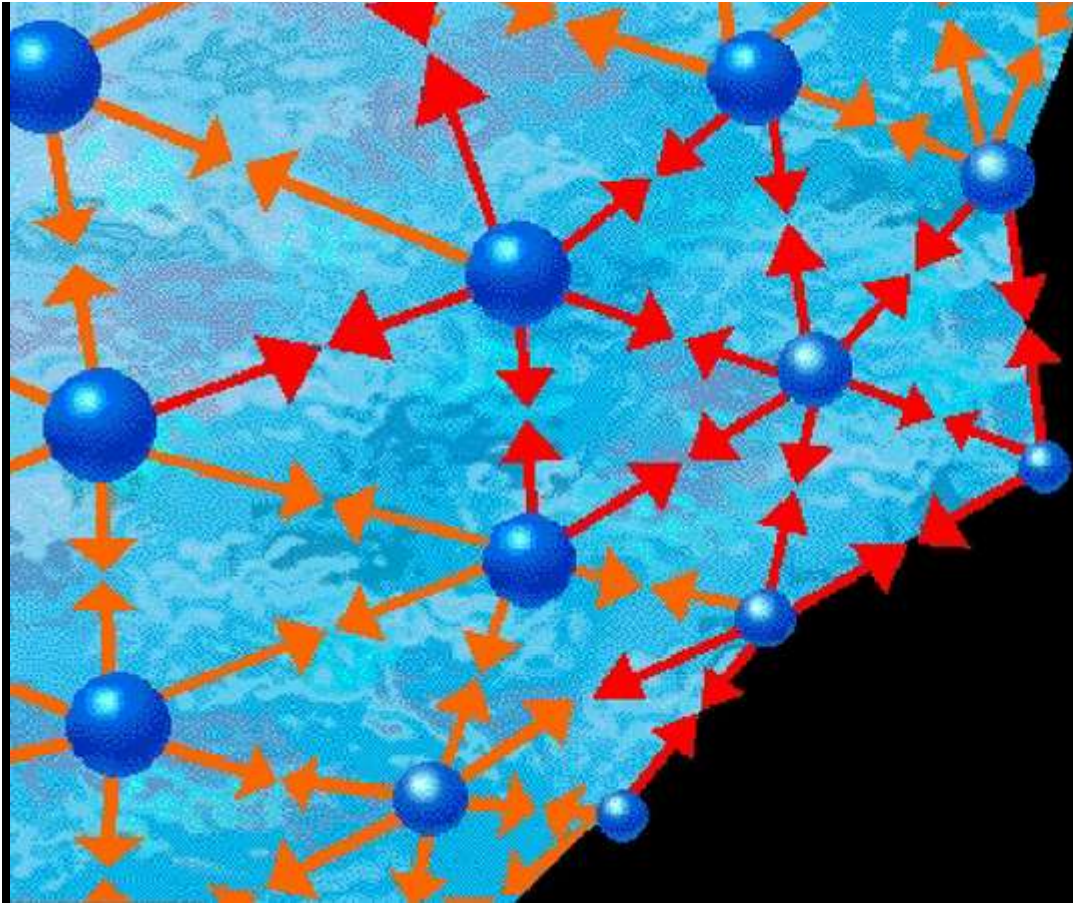


© 2012 Encyclopædia Britannica, Inc.

Figure 2: Three possible orientations of a uniform square prism floating in liquid of twice its density. The stable orientation is (C) (see text). *Encyclopædia Britannica, Inc.*

## Surface tension of liquids



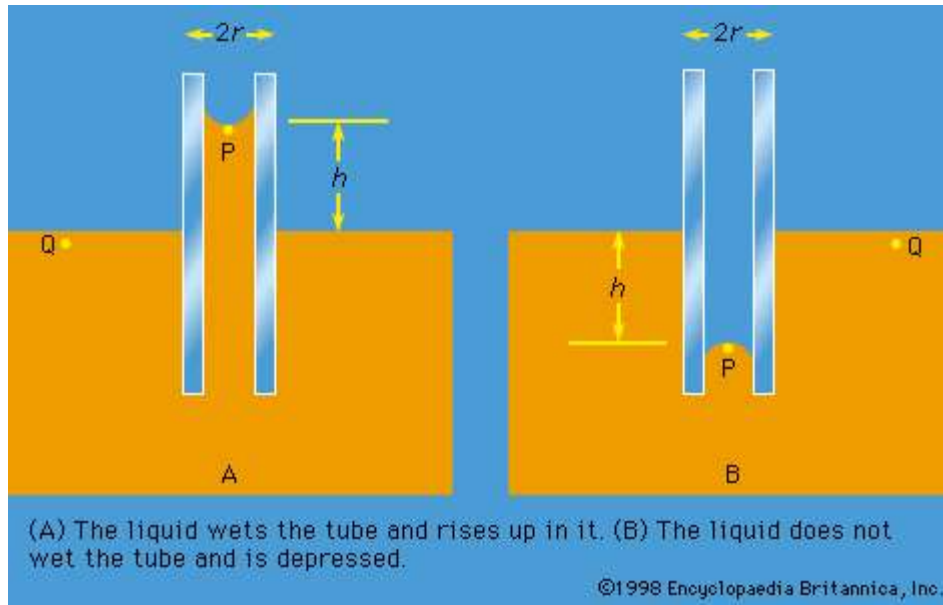


**surface tension** Explanation of surface tension. *Encyclopædia Britannica, Inc.*

Of the many hydrostatic phenomena in which the surface tension of liquids plays a role, the most significant is probably capillarity. Consider what happens when a tube of narrow bore, often called a capillary tube, is dipped into a liquid. If the liquid “wets” the tube (with zero contact angle), the liquid surface inside the tube forms a concave meniscus, which is a virtually spherical surface having the same radius,  $r$ , as the inside of the tube. The tube experiences a downward force of magnitude  $2\pi r d\sigma$ , where  $\sigma$  is the surface tension of the liquid, and the liquid experiences a reaction of equal magnitude that lifts the meniscus through a height  $h$  such that

$$2\pi r\sigma = \pi r^2 h \rho g \quad (127)$$

—*i.e.*, until the upward force for which surface tension is responsible is balanced by the weight of the column of liquid that has been lifted. If the liquid does not wet the tube, the meniscus is convex and depressed through the same distance  $h$  (see Figure 3). A simple method for determining surface tension involves the measurement of  $h$  in one or the other of these situations and the use of equation (127) thereafter.



$$2\pi r\sigma = \pi r^2 h \rho g \quad (127)$$

Figure 3: Capillarity. *Encyclopaedia Britannica, Inc.*

It follows from equations (124) and (127) that the pressure at a point P just below the meniscus differs from the pressure at Q by an amount

$$\rho gh = \frac{2\sigma}{r}; \quad (128)$$

it is less than the pressure at Q in the case to which [Figure 3A](#) refers and greater than the pressure at Q in the other case. Since the pressure at Q is just the atmospheric pressure, it is equal to the pressure at a point immediately above the meniscus. Hence, in both instances there is a pressure difference of  $2\sigma/r$  between the two sides of the curved meniscus, and in both the higher pressure is on the inner side of the curve. Such a pressure difference is a requirement of equilibrium wherever a liquid surface is curved. If the surface is curved but not spherical, the pressure difference is

$$\sigma(r_1^{-1} + r_2^{-1}), \quad (129)$$

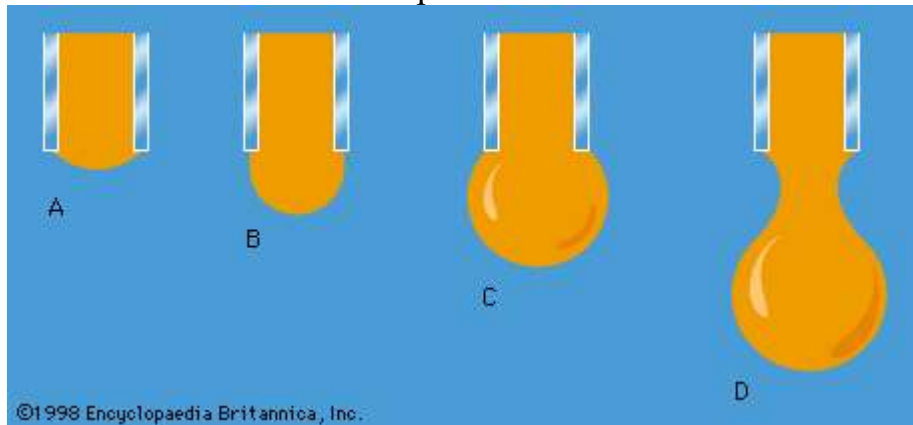
where  $r_1$  and  $r_2$  are the two principal radii of curvature. If it is cylindrical, one of these radii is infinite, and, if it is curved in opposite directions, then for the purposes of (129) they should be treated as being of opposite sign.

$$\sigma(r_1^{-1} + r_2^{-1}), \quad (129)$$

The diagrams in [Figure 3](#) were drawn to represent cross sections through cylindrical tubes, but they might equally well represent two vertical parallel plates that are partly submerged in the liquid a small distance apart. Consideration of how the pressure varies with height shows that over the range of height  $h$  the plates experience a greater pressure on their outer surfaces than on their inner surfaces; this is true whether the

liquid wets both plates or not. It is a matter of observation that small objects floating near one another on the surface of a liquid tend to move toward one another, and it is the pressure difference just referred to that makes them behave in this way.

One other problem having to do with surface tension will be considered here. The diagrams in [Figure 4](#) show stages in the growth of a liquid drop on the end of a tube which the liquid is supposed to wet. In passing from stage A to stage B, by which time the drop is roughly hemispheric in shape, the radius of curvature of the drop diminishes; and it follows from (128) that, to bring about this growth, one must slowly increase the pressure of the liquid inside the tube. If the pressure could be held steady at the value corresponding to B, the drop would then become unstable, because any further growth (*e.g.*, to the more or less spherical shape indicated in [Figure 4C](#)) would involve an increase in radius of curvature. The applied pressure would then exceed that required to hold the drop in equilibrium, and the drop would necessarily grow bigger still. In practice, however, it is easier to control the rate of flow of water through the tube, and hence the rate of growth of the drop, than it is to control the pressure. If the rate of flow is very small, drops will form the nonspherical shapes suggested by [Figure 4D](#) before they detach themselves and fall. It is not an easy matter to analyze the shape of a drop on the point of detachment, and there is no simple formula for the volume of the drop after it is detached.



©1998 Encyclopaedia Britannica, Inc.

$$\rho gh = \frac{2\sigma}{r}; \quad (128)$$

Figure 4: Stages in the formation of a liquid drop (see text). *Encyclopaedia Britannica, Inc.*