Hydrodynamics

Bernoulli’s law

Up to now the focus has been fluids at rest. This section deals with fluids that are in motion in a steady fashion such that the fluid velocity at each given point in space is not changing with time. Any flow pattern that is steady in this sense may be seen in terms of a set of streamlines, the trajectories of imaginary particles suspended in the fluid and carried along with it. In steady flow, the fluid is in motion but the streamlines are fixed. Where the streamlines crowd together, the fluid velocity is relatively high; where they open out, the fluid becomes relatively stagnant.

When Euler and Bernoulli were laying the foundations of hydrodynamics, they treated the fluid as an idealized inviscid substance in which, as in a fluid at rest in equilibrium, the shear stresses associated with viscosity are zero and the pressure *p* is isotropic. They arrived at a simple law relating the variation of *p* along a streamline to the variation of *v* (the principle is credited to Bernoulli, but Euler seems to have arrived at it first), which serves to explain many of the phenomena that real fluids in steady motion display. To the inevitable question of when and why it is justifiable to neglect viscosity, there is no single answer. Some answers will be provided later in this article, but other matters will be taken up first.

Consider a small element of fluid of mass *m*, which—apart from the force on it due to gravity—is acted on only by a pressure *p*. The latter is isotropic and does not vary with time but may vary from point to point in space. It is a well-known consequence of Newton’s laws of motion that, when a particle of mass *m* moves under the influence of its weight *mg* and an additional force *F* from a point P where its speed is v_P and its height is z_P to a point Q where its speed is v_Q and its height is z_Q, the work done by the additional force is equal to the increase in kinetic and potential energy of the particle—*i.e.,* that

$$\int_P^Q \boldsymbol{F} \cdot d\boldsymbol{s} = \left(\frac{1}{2}\right) m(v_Q^2 - v_P^2) + mg(z_Q - z_P). \qquad (130)$$

In the case of the fluid element under consideration, *F* may be related in a simple fashion to the gradient of the pressure, and one finds

$$\int_P^Q \boldsymbol{F} \cdot d\boldsymbol{s} = -m \int_P^Q \rho^{-1}\, dp. \qquad (131)$$

If the variations of fluid density along the streamline from P to Q are negligibly small, the factor ρ^{-1} may be taken outside the integral on the right-hand side of (131), which thereupon reduces to $\rho^{-1}(p_Q - p_P)$. Then (130) and (131) can be combined to obtain

- $$\int_P^Q \boldsymbol{F} \cdot ds = -m \int_P^Q \rho^{-1}\, dp. \qquad (131)$$
- $$\int_P^Q \boldsymbol{F} \cdot ds = \left(\frac{1}{2}\right) m(v_Q^2 - v_P^2) + mg(z_Q - z_P). \qquad (130)$$

$$\frac{p_P}{\rho} + \frac{v_P^2}{2} + gz_P = \frac{p_Q}{\rho} + \frac{v_Q^2}{2} + gz_Q. \qquad (132)$$

Since this applies for any two points that can be visited by a single element of fluid, one can immediately deduce Bernoulli's (or Euler's) important result that along each streamline in the steady flow of an inviscid fluid the quantity

$$\left(\frac{p}{\rho} + \frac{v^2}{2} + gz\right) \qquad (133)$$

is constant.

Under what circumstances are variations in the density negligibly small? When they are very small compared with the density itself—*i.e.,* when

$$\left(\frac{\Delta\rho}{\rho}\right) = \beta_S \Delta p = (\beta_S\, \rho)\Delta\left(\frac{v^2}{2} + gz\right) = \frac{\Delta\left(\frac{v^2}{2} + gz\right)}{V_S^2} << 1, \qquad (134)$$

where the symbol Δ is used to represent the extent of the change along a streamline of the quantity that follows it, and where V_s is the speed of sound (see below Compressible flow in gases). This condition is satisfied for all the flow problems having to do with water that are discussed below. If the fluid is air, it is adequately satisfied provided that the largest excursion in *z* is on the order of metres rather than kilometres and provided that the fluid velocity is everywhere less than about 100 metres per second.
Bernoulli's law indicates that, if an inviscid fluid is flowing along a pipe of varying cross section, then the pressure is relatively low at constrictions where the velocity is high and relatively high where the pipe opens out and the fluid stagnates. Many people find this situation paradoxical when they first encounter it. Surely, they say, a constriction should increase the local pressure rather than diminish it? The paradox evaporates as one learns to think of the pressure changes along the pipe as cause and the velocity

changes as effect, instead of the other way around; it is only because the pressure falls at a constriction that the pressure gradient upstream of the constriction has the right sign to make the fluid accelerate.

Paradoxical or not, predictions based on Bernoulli's law are well-verified by experiment. Try holding two sheets of paper so that they hang vertically two centimetres or so apart and blow downward so that there is a current of air between them. The sheets will be drawn together by the reduction in pressure associated with this current. Ships are drawn together for much the same reason if they are moving through the water in the same direction at the same speed with a small distance between them. In this case, the current results from the displacement of water by each ship's bow, which has to flow backward to fill the space created as the stern moves forward, and the current between the ships, to which they both contribute, is stronger than the current moving past their outer sides. As another simple experiment, listen to the hissing sound made by a tap that is almost, but not quite, turned off. What happens in this case is that the flow is so constricted and the velocity within the constriction so high that the pressure in the constriction is actually negative. Assisted by the dissolved gases that are normally present, the water cavitates as it passes through, and the noise that is heard is the sound of tiny bubbles collapsing as the water slows down and the pressure rises again on the other side.

Two practical devices that are used by hydraulic engineers to monitor the flow of liquids though pipes are based on Bernoulli's law. One is the venturi tube, a short length with a constriction in it of standard shape (see Figure 5A), which may be inserted into the pipe proper. If the velocity at point P, where the tube has a cross-sectional area A_P, is v_P and the velocity in the constriction, where the area is A_Q, is v_Q, the continuity condition—the condition that the mass flowing through the pipe per unit time has to be the same at all points along its length—suggests that $\rho_P A_P v_P = \rho_Q A_Q v_Q$, or that $A_P v_P = A_Q v_Q$ if the difference between ρ_P and ρ_Q is negligible. Then Bernoulli's law indicates

$$\rho g h = (p_P - p_Q) = \left(\frac{1}{2}\right) \rho v_P^2 \left[\left(\frac{A_P}{A_Q}\right)^2 - 1\right]. \qquad (135)$$

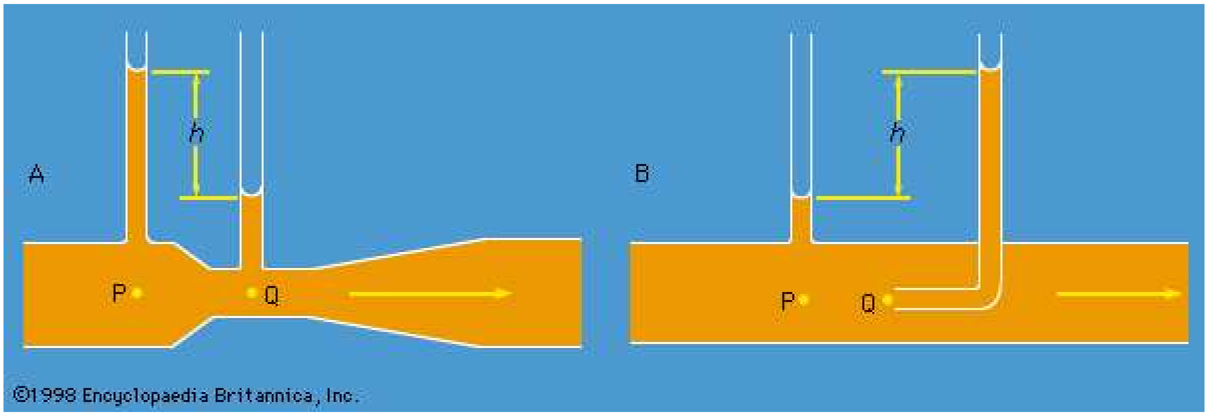


Figure 5: Schematic representation of (A) a venturi tube and of (B) a pitot tube.*Encyclopædia Britannica, Inc.*

Thus one should be able to find v_P, and hence the quantity Q (= $A_P v_P$) that engineers refer to as the rate of discharge, by measuring the difference of level h of the fluid in the two side tubes shown in the diagram. At low velocities the pressure difference (p_P - p_Q) is greatly affected by viscosity (see below Viscosity), and equation (135) is unreliable in consequence. The venturi tube is normally used, however, when the velocity is large enough for the flow to be turbulent (see below Turbulence). In such a circumstance, equation (135) predicts values for Q that agree with values measured by more direct means to within a few parts percent, even though the flow pattern is not really steady at all.

$$\rho g h = (p_P - p_Q) = \left(\frac{1}{2}\right) \rho v_P^2 \left[\left(\frac{A_P}{A_Q}\right)^2 - 1\right]. \qquad (135)$$

The other device is the pitot tube, which is illustrated in Figure 5B. The fluid streamlines divide as they approach the blunt end of this tube, and at the point marked Q in the diagram there is complete stagnation, since the fluid at this point is moving neither up nor down nor to the right. It follows immediately from Bernoulli's law that

$$\rho g h = (p_Q - p_P) = \left(\frac{1}{2}\right) \rho v_P^2. \qquad (136)$$

As with the venturi tube, one should therefore be able to find v_P from the level difference h.

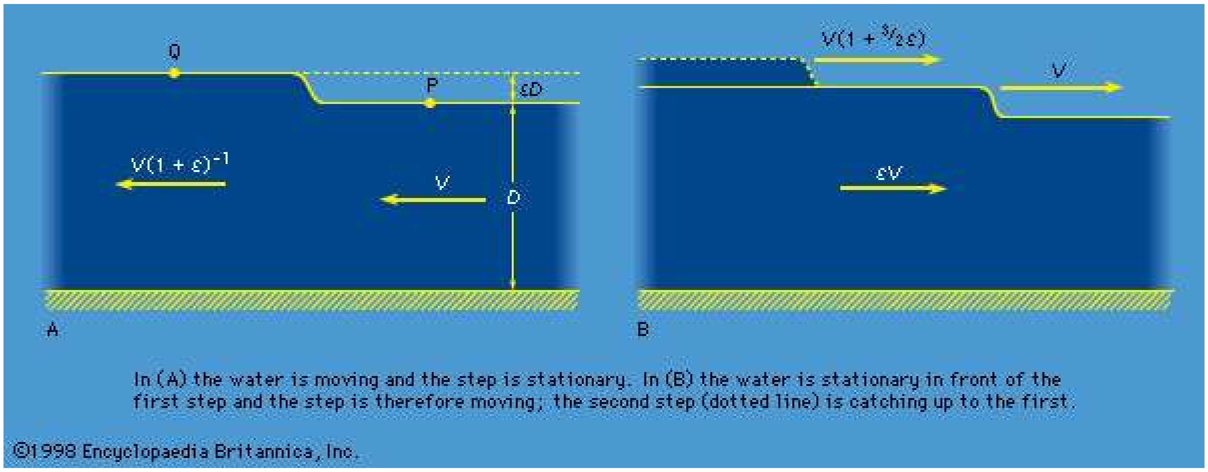
One other simple result deserves mention here. It concerns a jet of fluid emerging through a hole in the wall of a vessel filled with liquid under pressure. Observation of jets shows that after emerging they narrow slightly before settling down to a more or less uniform cross section known as the vena contracta. They do so because the streamlines are converging on the hole inside the vessel and are obliged to continue converging for a short while outside. It was Torricelli who first suggested that, if the pressure excess inside the vessel is generated by a head of liquid h, then the velocity v at the vena contracta is the velocity that a free particle would reach on falling through a height h—*i.e.,* that

$$v = \sqrt{(2gh)}. \qquad (137)$$

This result is an immediate consequence, for an inviscid fluid, of the principle of energy conservation that Bernoulli's law enshrines.
In the following section, Bernoulli's law is used in an indirect way to establish a formula for the speed at which disturbances travel over the surface of shallow water. The explanation of several interesting phenomena having to do with water waves is buried in this formula. Analogous phenomena dealing with sound waves in gases are discussed below in Compressible flow in gases, where an alternative form of Bernoulli's law is introduced. This form of the law is restricted to gases in steady flow but is not restricted to flow velocities that are much less than the speed of sound. The complication that viscosity represents is again ignored throughout these two sections.

Waves on shallow water

Imagine a layer of water with a flat base that has a small step on its surface, dividing a region in which the depth of the water is uniformly equal to D from a region in which it is uniformly equal to $D(1 + \varepsilon)$, with $\varepsilon \ll 1$. Let the water in the shallower region flow toward the step with some uniform speed V, as Figure 6A suggests, and let this speed be just sufficient to hold the step in the same position so that the flow pattern is a steady one.
The continuity condition (*i.e.,* the condition that as much water flows out to the left per unit time as flows in from the right) indicates that in the deeper region the speed of the water is $V(1 + \varepsilon)^{-1}$. Hence by applying Bernoulli's law to the points marked P and Q in the diagram, which lie on the same streamline and at both of which the pressure is atmospheric, one may deduce that



In (A) the water is moving and the step is stationary. In (B) the water is stationary in front of the first step and the step is therefore moving; the second step (dotted line) is catching up to the first.

Figure 6: Steps on the surface of shallow water.*Encyclopædia Britannica, Inc.*

$$g\varepsilon D = \left(\frac{1}{2}\right) V^2[1 - (1+\varepsilon)^{-2}] \approx \varepsilon V^2$$

— i.e., that

$$V = \sqrt{(gD)}. \qquad (138)$$

This result shows that, if the water in the shallower region is in fact stationary (see Figure 6B), the step advances over it with the speed *V* that equation (138) describes, and it reveals incidentally that behind the step the deeper water follows up with speed $V[1 - (1 + \varepsilon)^{-1}] \approx \varepsilon V$. The argument may readily be extended to disturbances of the surface that are undulatory rather than steplike. Provided that the distance between successive crests—a distance known as the wavelength and denoted by λ—is much greater than the depth of the water, *D*, and provided that its amplitude is very much less than *D*, a wave travels over stationary water at a speed given by (138). Because their speed does not depend on wavelength, the waves are said to be nondispersive.

$$g\varepsilon D = \left(\frac{1}{2}\right) V^2[1 - (1+\varepsilon)^{-2}] \approx \varepsilon V^2$$

— i.e., that

$$V = \sqrt{(gD)}. \qquad (138)$$

Evidently waves that are approaching a shelving beach should slow down as *D* diminishes. If they are approaching it at an angle, the slowing-down effect bends, or refracts, the wave crests so that they are nearly parallel to the shore by the time they ultimately break.

Suppose now that a small step of height ε*D* (ε << 1) is traveling over stationary water of uniform depth *D* and that behind it is a second step of much the same height traveling in the same direction. Because the second step (suggested by a dotted line in Figure 6B) is traveling on a base that is moving at εSquare root of$\sqrt{(gD)}$ and because the thickness of that base is (1

+ ε)D rather than D, the speed of the second step is approximately (1 + 3ε/2)Square root of√(gD). Since this is greater than Square root of√(gD), the second step is bound to catch up with the first. Hence, if there are a succession of infinitesimal steps that raise the depth continuously from D to some value D', which differs significantly from D, then the ramp on the surface is bound to become steeper as it advances. It may be shown that if D' exceeds about 1.3D, the ramp ultimately becomes a vertical step of finite height and that the step then "breaks." A finite step that has broken dissipates energy as heat in the resultant foaming motion, and Bernoulli's equation is no longer applicable to it. A simple argument based on conservation of momentum rather than energy, however, suffices to show that its velocity of propagation is

$$\sqrt{\left(\frac{gD'(D'+D)}{2D}\right)}. \qquad (139)$$

Tidal bores, which may be observed on some estuaries, are examples on the large scale of the sort of phenomena to which (139) applies. Examples on a smaller scale include the hydraulic jumps that are commonly seen below weirs and sluice gates where a smooth stream of water suddenly rises at a foaming front. In this case, (139) describes the speed of the water, since the front itself is more or less stationary.

$$\sqrt{\left(\frac{gD'(D'+D)}{2D}\right)}. \qquad (139)$$

When water is shallow but not extremely shallow, so that correction terms of the order of $(D/\lambda)^2$ are significant, waves of small amplitude become slightly dispersive (see below Waves on deep water). In this case, a localized disturbance on the surface of a river or canal, which is guided by the banks in such a way that it can propagate in one direction only, is liable to spread as it propagates. If its amplitude is not small, however, the tendency to spread due to dispersion may in special circumstances be subtly balanced by the factors that cause waves of relatively large amplitude to form bores, and the result is a localized hump in the surface, of symmetrical shape, which does not spread at all. The phenomenon was first observed on a canal near Edinburgh in 1834 by a Scottish engineer named Scott Russell; he later wrote a graphic account of following on horseback, for well over a kilometre, a "large solitary elevation . . . which continued its course along the channel apparently without change of form." What Scott Russell saw is now called a soliton Solitons on canals can have various widths, but the smaller the width the. larger the height must be and the faster the soliton travels. Thus, if a high, narrow soliton is formed behind a low, broad one, it will catch up with the low one. It turns out that, when the high soliton does so, it passes through the low one and emerges with its shape unchanged (see Figure 7).

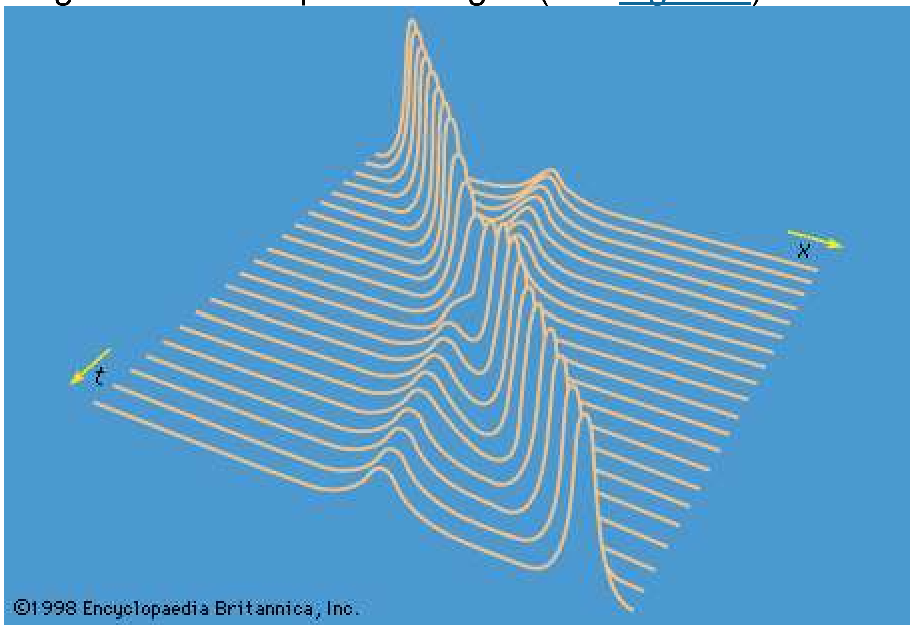


Figure 7: Interaction of two solitons (see text).*Encyclopædia Britannica, Inc.*

It is now recognized that many of the nonlinear differential equations that appear in diverse branches of physics have solutions of large amplitude corresponding to solitons and that the remarkable capacity of solitons for surviving encounters with other solitons is universal. This discovery has stimulated much interest among mathematicians and physicists, and understanding of solitons is expanding rapidly.

Compressible flow in gases

Compressible flow refers to flow at velocities that are comparable to, or exceed, the speed of sound. The compressibility is relevant because at such velocities the variations in density that occur as the fluid moves from place to place cannot be ignored.

Suppose that the fluid is a gas at a low enough pressure for the ideal equation of state, equation (118), to apply and that its thermal conductivity is so poor that the compressions and rarefactions undergone by each element of the gas may be treated as adiabatic (see above). In this case, it follows from equation (120) that the change of density accompanying any small change in pressure, *dp*, is such that

$$T \propto \rho^{(\gamma-1)}, \quad p \propto \rho^{\gamma}, \qquad (120)$$

$$\rho^{-1} dp = \left(\frac{\gamma}{\gamma-1}\right) d\left(\frac{p}{\rho}\right). \qquad (140)$$

This makes it possible to integrate the right-hand side of equation (131), and one thereby arrives at a version of Bernoulli's law for a steady compressible flow of gases which states that

$$\left(\frac{\gamma p}{(\gamma-1)\rho}\right)+\frac{v^2}{2}+gz \qquad (141)$$

is constant along a streamline. An equivalent statement is that

$$\frac{C_p T}{M}+\frac{v^2}{2}+gz \qquad (142)$$

is constant along a streamline. It is worth noting that, when a gas flows through a nozzle or through a shock front (see below), the flow, though adiabatic, may not be reversible in the thermodynamic sense. Thus the entropy of the gas is not necessarily constant in such flow, and as a consequence the application of equation (120) is open to question. Fortunately, the result expressed by (141) or (142) can be established by arguments that do not involve integration of (131). It is valid for steady adiabatic flow whether this is reversible or not.

- $$\left(\frac{\gamma p}{(\gamma-1)\rho}\right)+\frac{v^2}{2}+gz \qquad (141)$$
- $$\frac{C_p T}{M}+\frac{v^2}{2}+gz \qquad (142)$$

Bernoulli's law in the form of (142) may be used to estimate the variation of temperature with height in the Earth's atmosphere. Even on the calmest day the atmosphere is normally in motion because convection currents (see below Convection) are set up by heat derived from sunlight that is released at the Earth's surface. The currents are indeed adiabatic to a good approximation, and their velocity is generally small enough for the term v^2 in (142) to be negligible. One can therefore deduce without more ado that the temperature of the atmosphere should fall off in a linear fashion—*i.e.,* that

$$T(z)=T(0)-\beta z=T(0)-\left(\frac{Mg}{C_p}\right)z. \qquad (143)$$

Here β is used to represent the temperature lapse rate, and the value suggested for this quantity, (Mg/C_p), is close to 10° C per kilometre for dry air. This prediction is not exactly fulfilled in practice. Within the troposphere (*i.e.,* to the heights of about 10 kilometres to which convection currents extend), the mean temperature does decrease with height in a linear fashion, but β is only about 6.5° C per kilometre. It is the water vapour in the atmosphere, which condenses as the air rises and cools, that lowers the lapse rate to this value by increasing the effective value of C_p. The fact that the lapse rate is smaller for moist air than for dry air means that a stream of moist air which passes over a mountain range and which deposits its moisture as rain or snow at the summit is warmer when it descends to sea level on the other side of the range than it was when it started. The foehn wind of the Alps owes its warmth to this effect.

The variation of the pressure of the atmosphere with height may be estimated in terms of β, using the equation

$$p(z) = p(0)\left[1 - \frac{\beta z}{T(0)}\right]^{Mg/R\beta}. \qquad (144)$$

This is obtained by integration of (123), using (118) and (143).

- $$\frac{dp}{dz} = -\rho g. \qquad (123)$$
- $$T(z) = T(0) - \beta z = T(0) - \left(\frac{Mg}{C_p}\right)z. \qquad (143)$$

In the form of equation (141), Bernoulli's law may be used to calculate the speed of sound in gases. The argument is directly analogous to the one applied in the previous section to waves on shallow water—and, indeed, the diagrams in Figure 6 can serve to illustrate the argument here too, if they are regarded as plots of gas density (or else of pressure or temperature, which go hand in hand with density in adiabatic flow) versus position. The results of the argument will be stated without proof. If there exists an infinitesimal step in the density of the gas, it will remain stationary provided that the gas flows uniformly through it toward the region of higher density, with a velocity

$$V_s = \sqrt{\left(\frac{\gamma p}{\rho}\right)}. \qquad (145)$$

If the gas is stationary, then (145) describes the velocity with which the step moves. It also describes the speed of propagation of the sort of undulatory variation of density that constitutes a sound wave of fixed frequency or pitch. Because the speed of sound is independent of pitch, sound waves, like waves on shallow water, are nondispersive. This is just as well. It is only because there is no dispersion that one can understand the words of a distant speaker or listen to a symphony orchestra with pleasure from the back of an auditorium as well as from the front.

It should be noted that the formula for the speed of sound in gases may be proved in other ways, and Newton came close to it a century before

Bernoulli's time. However, because Newton failed to appreciate the distinction between adiabatic and isothermal flow, his answer lacked the factor γ occurring in (145). The first person to correct this error was Pierre-Simon Laplace.

The above statements apply to density steps or undulations, the amplitude of which is infinitesimal, and they need some modification if the amplitude is large. In the first place it is found, as for waves on shallow water and for very much the same reasons, that, where two small density steps are moving parallel to one another, the second is bound to catch up with the first. It follows that, if there exists a propagating region in which the density rises in a continuous fashion from ρ to ρ′, where (ρ′ - ρ) is not necessarily small, then the width of this region is bound to diminish as time passes. Ultimately a shock front develops over which the density—and hence the pressure and temperature—rises almost discontinuously. There are processes within the shock front, vaguely analogous on the molecular scale to the foaming of a breaking water wave, by which energy is dissipated as heat. The speed of propagation, V_{sh}, of a shock front in a gas that is stationary in front of it may be expressed in terms of V_s and V_s', the velocities of small-amplitude sound waves in front of the shock and behind it, respectively, by the equation

$$2V_{sh}^2 = \left[\frac{(\gamma + 1)\rho'}{\gamma\rho}\right]V_s'^2 + V_s^2. \qquad (146)$$

Thus, if the shock is a strong one (ρ′ >> ρ), V_{sh} may be significantly greater than both V_s and V_s'.

Even the gentlest sound wave, in which density and pressure initially oscillate in a smooth and sinusoidal fashion, develops into a succession of weak shock fronts in time. More noticeable shock fronts are a feature of the flow of gases at supersonic speeds through the nozzles of jet engines and accompany projectiles that are moving through stationary air at supersonic speeds. In certain circumstances when a supersonic aircraft is following a curved path, the accompanying shock wave may accidentally reinforce itself in places and thereby become offensively noticeable as a "sonic boom," which may break windowpanes and cause other damage. Strong shock fronts also occur immediately after explosions, of course, and when windowpanes are broken by an explosion, the broken glass tends to fall outward rather than inward. Such is the case because the glass is sucked out by the relatively low density and pressure that succeed the shock itself.

The diagrams in Figure 8 show a well-known construction attributed to the Austrian physicist Ernst Mach that explains the origin of the shock front accompanying a supersonic projectile. The circular arcs in this figure represent cross sections through spherical disturbances that are spreading with speed V_s from centres (S′, S″, etc.), which mark the position of the moving source S at the time when they were emitted. If the source is something like the tip of an arrow, which disturbs the air by parting it as it travels along but which is inaudible when stationary, then each “disturbance” due to some infinitesimal displacement of the tip is a spherical shell of infinitesimal thickness within which a small radial velocity has been imparted to the air. There is an infinite number of such disturbances, overlapping one another, of which only a handful are represented in Figure 8. When the velocity of the source, U, is less than V_s (Figure 8A), the result of adding them together is the sort of steady backflow that is to be expected around a moving obstacle, and there is no sound emission in the normal sense; the source remains inaudible. When U exceeds V_s, however, the spherical disturbances reinforce one another, as Figure 8B shows, on a conical caustic surface, which makes an angle of $\sin^{-1}(U/V)$ to the line of travel of the source, and it is on this surface that a shock front is to be expected. The cone becomes sharper as the source speeds up.

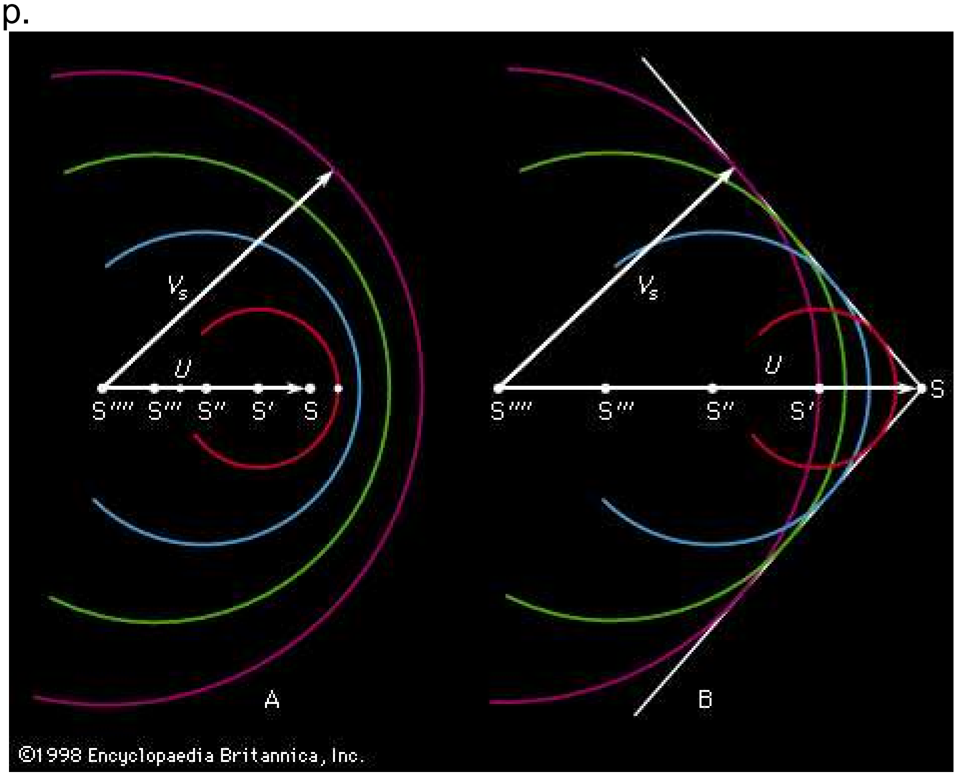


Figure 8: *Mach's construction.* (A) Source speed U less than speed of sound V_s, (B) U greater than V_s (see text).*Encyclopædia Britannica, Inc.*