## **Viscosity**

As shown above, a number of phenomena of considerable physical interest can be discussed using little more than the <u>law of conservation</u> of energy, as expressed by <u>Bernoulli's law</u>. However, the argument has so far been restricted to cases of steady flow. To discuss cases in which the flow is not steady, an <u>equation of motion</u> for fluids is needed, and one cannot write down a realistic equation of motion without facing up to the problems presented by viscosity, which have so far been deliberately set aside.

## Stresses in laminar motion

The concept of viscosity was first formalized by Newton, who considered the shear stresses likely to arise when a fluid undergoes what is called laminar motion with the sort of velocity profile that is suggested in Figure 9A; the laminae here are planes normal to the  $x_2$ -axis, and they are moving in the direction of the  $x_1$ -axis with a velocity  $v_1$ , which increases in a linear fashion with  $x_2$ . Newton suggested that, as each lamina slips over the one below, it exerts a sort of frictional force upon the latter in the forward direction, in which case the upper lamina is bound to experience an equal reaction in the backward direction. The strength of these forces per unit area <u>constitutes</u> the component of <u>shear stress</u> normally written as  $\sigma_{12}$  (not to be confused with surface tension, for which the symbol  $\sigma$  has been used above). Figure 9B shows, in elevation, an enlarged view of an infinitesimal element of the fluid of cubic shape, and the directions of the forces experienced by this cube associated with  $\sigma_{12}$  are indicated by arrows. Other arrows show the directions of the forces associated with the so-called normal stresses  $\sigma_{11}$  and  $\sigma_{22}$ , which in the absence of motion of the fluid would both be equal, by <u>Pascal's law</u>, to -p. Now  $\sigma_{12}$  is clearly zero when the rate of variation of velocity,  $\partial v_1 / \partial x_2$ , is zero, for then there is no slip, and presumably it increases monotonically as  $\partial v_1 / \partial x_2$  increases. Newton made the plausible assumption that the two are linearly related—*i.e.*, that



Figure 9: Laminar motion and associated stresses. Encyclopædia Britannica, Inc.

$$\sigma_{12} = \eta \left(\frac{\partial v_1}{\partial x_2}\right). \quad (147)$$

The full name for the coefficient  $\eta$  is <u>shear viscosity</u> to distinguish it from the bulk viscosity, *b*, which is defined below. The word shear, however, is frequently omitted in this <u>context</u>.

Now if the only shear stress acting on the cubic element of fluid sketched in Figure <u>9B</u> were  $\sigma_{12}$ , the cube would experience a torque tending to make it twist in a clockwise sense. Since the magnitude of the torque would vary like the third power of the linear dimensions of the cube, whereas the moment of inertia of the element would vary like the fifth power, the resultant angular acceleration for an infinitesimal cube would be infinite. One may infer that any tendency to twist in a clockwise sense gives rise instantaneously to an additional shear stress  $\sigma_{21}$ , the direction of which is indicated in the diagram, and that  $\sigma_{12}$  and  $\sigma_{21}$  are equal at all times. It follows that equation (<u>147</u>) cannot be a complete expression for these shear stresses, for it does not include the possibility that the fluid is moving in the  $x_2$  direction, with a velocity  $v_2$  that varies with  $x_1$ . The complete expression for what is called a Newtonian fluid is

$$\sigma_{12} = \eta \left(\frac{\partial v_1}{\partial x_2}\right). \quad (147)$$

$$\sigma_{12} = \sigma_{21} = \eta \left[ \left( \frac{\partial v_1}{\partial x_2} \right) + \left( \frac{\partial v_2}{\partial x_1} \right) \right].$$
(148)

Similar expressions may be written down for  $\sigma_{23}$  (=  $\sigma_{32}$ ) and  $\sigma_{31}$  (=  $\sigma_{13}$ ). Since Newton's day these <u>hypothetical</u> expressions have been fully <u>substantiated</u> for gases and simple liquids, not only by experiment but also by analysis of the molecular motions and molecular interactions in such fluids undergoing shear, and for such fluids one can even predict the magnitude of  $\eta$  with reasonable success. There do exist, however, more complicated fluids for which the Newtonian description of shear stress is inadequate, and some of these are very familiar in the home. In the whites of eggs, for example, and in most <u>shampoos</u>, there are long-chain molecules that become entangled with one another, and entanglement may hinder their efforts to respond to changes of <u>environment</u> associated with flow. As a result, the stresses acting in such fluids may reflect the deformations experienced by the fluid in the recent past as much as the instantaneous rate of deformation. Moreover, the relation between <u>stress</u> and rate of deformation may be far from linear. Non-Newtonian effects, interesting though they are, lie outside the scope of the present discussion, however.

The sort of velocity profile that is suggested by Figure 9B may be established by containing the fluid between two parallel flat plates and moving one plate relative to the other. The possibility exists that in this situation the layers of fluid immediately in contact with each plate will slip over them with some finite velocity (indicated in the diagram by an arrow labeled  $v_{slip}$ ). If so, the frictional stresses associated with this slip must be such as to balance the shear stress  $\eta(\partial v_1/\partial x_2)$  exerted on each of these layers by the rest of the fluid. Little is known about fluid-solid frictional stresses, but intelligent guesswork suggests that they are proportional in magnitude to  $v_{slip}$  and that, in the circumstances to which Figure 9A refers, the distance d below the surface of the stationary bottom plate at which the straight line representing the variation of  $v_1$  with  $x_2$  extrapolates to zero should be of the same order of magnitude as the diameter of a molecule if the fluid is a liquid or as the molecular "mean free path" if it is a gas. These distances are normally very small compared with the separation of the plates, D. Accordingly, fluid flow patterns may normally be treated as subject to the boundary condition that at a fluid-solid interface the relative velocity of the fluid is zero. No reliable evidence for failure of predictions based on this no-slip boundary condition has yet been found, except in the case of what is called Knudsen flow of gases (*i.e.*, flow at such low pressures that the mean free path is comparable in length with the dimensions of the apparatus).

If a fluid is flowing steadily between two parallel plates that are both stationary and if its velocity must be zero in contact with both of them, the velocity profile must necessarily have the form indicated in Figure 10. A force in the forward direction due to the shear stress  $\eta(\partial v_1/\partial x_2)$  is transmitted to the plates, and an equal force in the backward direction acts on the fluid. The motion therefore cannot be maintained unless the <u>pressure</u> acting on the fluid is greater on the left of the diagram than it is on the right. A full analysis shows the velocity profile to be parabolic, and it indicates that the rate of discharge is related to the pressure <u>gradient</u> by the equation



Figure 10: Velocity profile for laminar flow between two plates (or inside a cylindrical tube), driven by a pressure gradient (see text).*Encyclopædia Britannica, Inc.* 

$$Q = -\left(\frac{WD^3}{12\eta}\right)\left(\frac{dp}{dx_1}\right),\qquad(149)$$

where W( >> D) is the width of the plates, measured perpendicular to the diagram in <u>Figure 10</u>. A similar analysis of the problem of steady flow through a (horizontal) cylindrical pipe of uniform diameter *D*, to which <u>Figure 10</u> could equally well apply, shows the rate of discharge in this case to be given by

$$Q = -\left(\frac{\pi D^4}{128\eta}\right) \left(\frac{dp}{dx_1}\right); \qquad (150)$$

this famous result is known as <u>Poiseuille's equation</u>, and the type of flow to which it refers is called Poiseuille flow.

## **Bulk viscosity**

Viscosity may affect the normal stress components,  $\sigma_{11}$ ,  $\sigma_{22}$ , and  $\sigma_{33}$ , as well as the shear stress components. To see why this is so, one needs to examine the way in which stress components transform when one's reference axes are rotated. Here, the result will be stated without proof that the general expression for  $\sigma_{11}$  consistent with (<u>148</u>) is

$$\sigma_{12} = \sigma_{21} = \eta \left[ \left( \frac{\partial v_1}{\partial x_2} \right) + \left( \frac{\partial v_2}{\partial x_1} \right) \right].$$
(148)

$$\sigma_{11} = -p + \left(b + \frac{4\eta}{3}\right) \left(\frac{\partial v_1}{\partial x_1}\right) + \left(b - \frac{2\eta}{3}\right) \left(\frac{\partial v_2}{\partial x_2}\right) + \left(b - \frac{2\eta}{3}\right) \left(\frac{\partial v_3}{\partial x_3}\right).$$
(151)

On the right-hand side of this equation, p represents the <u>equilibrium</u> pressure defined in terms of local density and <u>temperature</u> by the <u>equation of state</u>, and b is another viscosity coefficient known as the <u>bulk viscosity</u>.

The bulk viscosity is relevant only where the density is changing. Thus it plays a role in <u>attenuating</u> sound waves in fluids and may be estimated from the magnitude of the attenuation. If the fluid is effectively incompressible, however, so that changes of <u>density</u> may be ignored, the flow is everywhere subject to the <u>continuity</u> condition that

$$\left(\frac{\partial v_1}{\partial x_1}\right) + \left(\frac{\partial v_2}{\partial x_2}\right) + \left(\frac{\partial v_3}{\partial x_3}\right) [\equiv \nabla \cdot \boldsymbol{v} \text{ or div } \boldsymbol{v}] = 0.$$
(152)

The terms in (151) that involve b then cancel, and the expression simplifies to

$$\sigma_{11} = -p + \left(b + \frac{4\eta}{3}\right) \left(\frac{\partial v_1}{\partial x_1}\right) + \left(b - \frac{2\eta}{3}\right) \left(\frac{\partial v_2}{\partial x_2}\right) + \left(b - \frac{2\eta}{3}\right) \left(\frac{\partial v_3}{\partial x_3}\right),$$

$$\sigma_{11} = -p + 2\eta \left(\frac{\partial v_1}{\partial x_1}\right).$$
(153)
(153)

Similar equations may be written down for  $\sigma_{22}$  and  $\sigma_{33}$ . These simpler expressions provide the basis for the argument that follows, and the bulk viscosity can be left on one side.

## Measurement of shear viscosity

A variety of methods are available for the measurement of shear viscosity. One standard method involves measurement of the pressure gradient along a pipe for various rates of flow and application of Poiseuille's equation. Other methods involve measurement either of the damping of the torsional oscillations of a solid disk supported between two parallel plates when fluid is admitted to the space between the plates, or of the effect of the fluid on the frequency of the oscillations.

The Couette viscometer deserves a fuller explanation. In this device, the fluid occupies the space between two coaxial cylinders of radii *a* and *b* (> *a*); the outer cylinder is rotated with uniform <u>angular velocity</u>  $\omega_0$ , and the resultant torque transmitted to the inner stationary cylinder is measured. If both the terms on the right-hand side of equation (<u>148</u>) are taken into account, the shear stress in the circulating

fluid is found to be proportional to  $r(d\omega/dr)$  rather than to (dv/dr)—not an unexpected result, since it is only if  $\omega$ , the angular velocity of the fluid, varies with radius r that there is any slip between one cylindrical lamina of fluid and the next. The torque transmitted through the fluid is therefore proportional to  $r^3(d\omega/dr)$ . In the steady state, the opposing torques acting on the inner and outer surfaces of each cylindrical lamina of fluid must be of equal magnitude—otherwise the laminae accelerate—and this means that  $r^3(d\omega/dr)$  must be independent of r. There are two basic modes of motion for a circulating fluid that satisfy this condition: in one, the liquid rotates as a solid body would, with an angular velocity that does not vary with r, and the torque is everywhere zero; in the other,  $\omega$  varies like  $r^{-2}$ . The angular velocity of the fluid in a Couette viscometer can be viewed as a mixture of these two modes in proportions that satisfy the boundary conditions at r = a and r = b. The torque transmitted per unit length of the cylinders turns out to be given by

 $4\pi\eta\omega_0 \left[\frac{b^2 a^2}{(b^2 - a^2)}\right].$  (154)

It may be added that if the inner cylinder is absent, the steady flow pattern consists only of the first mode—*i.e.*, the fluid rotates like a solid body with uniform angular velocity  $\omega_0$ . If the outer cylinder is absent, however, and the inner one rotates, it then consists only of the second mode. The angular velocity falls off like  $r^{-2}$ , and the velocity v falls off like  $r^{-1}$ .

In the equation of motion given in the following section, the shear viscosity occurs only in the combination ( $\eta/\rho$ ). This combination occurs so frequently in arguments of fluid <u>dynamics</u> that it has been given a special name—kinetic viscosity. The kinetic viscosity at normal temperatures and pressures is about  $10^{-6}$  square metre per second for water and about  $1.5 \times 10^{-5}$  square metre per second for air.