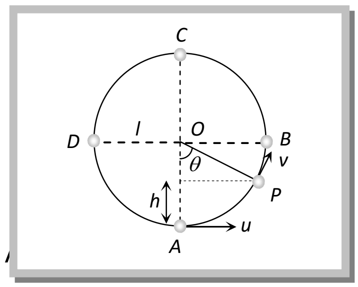
Motion in Vertical Circle.

This is an example of non-uniform circular motion. In this motion body is under the influence of gravity of earth. When body moves from lowest point to highest point. Its speed decrease and becomes minimum at highest point. Total mechanical energy of the body remains conserved and *KE* converts into *PE* and vice versa.

(1) **Velocity at any point on vertical loop:** If u is the initial velocity imparted to body at lowest point then. Velocity of body at height h is given by

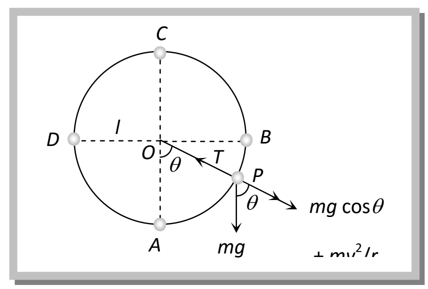


$$v = \sqrt{u^2 - 2gh} = \sqrt{u^2 - 2gl(1-\cos\theta)} \qquad [\text{As } h = l - l\cos\theta = l(1-\cos\theta)]$$

Where l in the length of the string

(2) **Tension at any point on vertical loop**: Tension at general point P, Newton's second law of motion.

Net force towards center = centripetal force



$$T - mg\cos\theta = \frac{mv^2}{l} \quad \text{Or} \quad T = mg\cos\theta + \frac{mv^2}{l}$$

$$T = \frac{m}{l}[u^2 - gl(2 - 3\cos\theta)] \qquad [\text{As } v = \sqrt{u^2 - 2gl(1-\cos\theta)}\,]$$

(3) **Velocity and tension in a vertical loop at different positions**

Position	Angle	Velocity	Tension
A	0°	*u*	$\frac{mu^2}{l} + mg$
B	90°	$\sqrt{u^2 - 2gl}$	$\frac{mu^2}{l} - 2mg$
C	180°	$\sqrt{u^2 - 4gl}$	$\frac{mu^2}{l} - 5mg$
D	270°	$\sqrt{u^2 - 2gl}$	$\frac{mu^2}{l} - 2mg$

It is clear from the table that: $T_A > T_B > T_C$ and $T_B = T_D$

$$T_A - T_B = 3mg,$$

$$T_A - T_C = 6mg$$

and $$T_B - T_C = 3mg$$

(4) **Various conditions for vertical motion:**

Velocity at lowest point	**Condition**
$u_A > \sqrt{5gl}$	Tension in the string will not be zero at any of the point and body will continue the circular motion.
$u_A = \sqrt{5gl},$	Tension at highest point C will be zero and body will just complete the circle.
$\sqrt{2gl} < u_A < \sqrt{5gl},$	Particle will not follow circular motion. Tension in string become zero somewhere between points *B* and *C* whereas velocity remain positive. Particle leaves circular path and follow parabolic trajectory.
$u_A = \sqrt{2gl}$	Both velocity and tension in the string becomes zero between *A* and *B* and particle will oscillate along semi-circular path.
$u_A < \sqrt{2gl}$	velocity of particle becomes zero between *A* and *B* but tension will not be zero and the particle will oscillate about the point *A*.

Note: *K.E.* of a body moving in horizontal circle is same throughout the path but the *K.E.* of the body moving in vertical circle is different at different places.

If body of mass *m* is tied to a string of length *l* and is projected with a horizontal velocity *u* then:

Height at which the velocity vanishes is $h = \frac{u^2}{2g}$

Height at which the tension vanishes is $h = \frac{u^2 + gl}{3g}$

(5) **Critical condition for vertical looping:** If the tension at *C* is zero, then body will just complete revolution in the vertical circle. This state of body is known as critical state. The speed of body in critical state is called as critical speed.

From the above table $\quad T_C = \frac{mu^2}{l} - 5mg = 0 \Rightarrow u = \sqrt{5gl}$

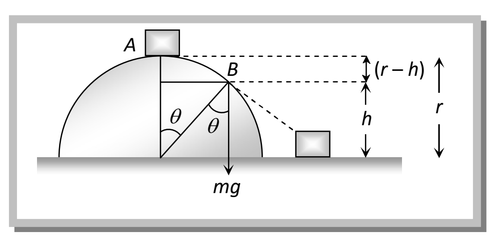
It means to complete the vertical circle the body must be projected with minimum velocity of $\sqrt{5gl}$ at the lowest point.

(6) **Various quantities for a critical condition in a vertical loop at different positions :**

Quantity	Point *A*	Point *B*	Point *C*	Point *D*	Point *P*
Linear velocity (*v*)	$\sqrt{5gl}$	$\sqrt{3gl}$	$\sqrt{gl}$	$\sqrt{3gl}$	$\sqrt{gl(3+2\cos\theta)}$
Angular velocity (ω)	$\sqrt{\frac{5g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}(3+2\cos\theta)}$
Tension in String (*T*)	6 *mg*	3 *mg*	0	3 *mg*	$3mg\,(1+\cos\theta)$
Kinetic Energy (*KE*)	$\frac{5}{2}mgl$	$\frac{3}{2}mgl$	$\frac{1}{2}mgl$	$\frac{3}{2}mgl$	$\frac{mgl}{2}(3+2\cos\theta)$
Potential Energy (*PE*)	0	*mgl*	2 *mgl*	*mgl*	$mgl\,(1-\cos\theta)$
Total Energy (*TE*)	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$

(7) **Motion of a block on frictionless hemisphere**:A small block of mass *m* slides down from the top of a frictionless hemisphere of radius *r*. The component of the force of gravity (*mg* cosθ) provides required centripetal force but at point *B* it's circular motion ceases and the block lose contact with the surface of the sphere.

For point *B*, by equating the forces, $mg\cos\theta = \frac{mv^2}{r}$ .....(i)



For point A and B, by law of conservation of energy

Total energy at point A = Total energy at point B

K.E.$_{(A)}$ + P.E.$_{(A)}$ = K.E.$_{(B)}$ + P.E.$_{(B)}$

$$0 + mgr = \frac{1}{2}mv^2 + mgh \Rightarrow v = \sqrt{2g(r-h)} \quad .....(ii)$$

and from the given figure $h = r\cos\theta$ .....(iii)

By substituting the value of v and h from eq^n (ii) and (iii) in eq^n (i)

$$mg\left(\frac{h}{r}\right) = \frac{m}{r}\left(\sqrt{2g(r-h)}\right)^2$$

$$\Rightarrow h = 2(r-h) \Rightarrow h = \frac{2}{3}r$$

i.e. the block lose contact at the height of $\frac{2}{3}r$ from the ground.

and angle from the vertical can be given by $\cos\theta = \frac{h}{r} = \frac{2}{3} \therefore \theta = \cos^{-1}\frac{2}{3}$.