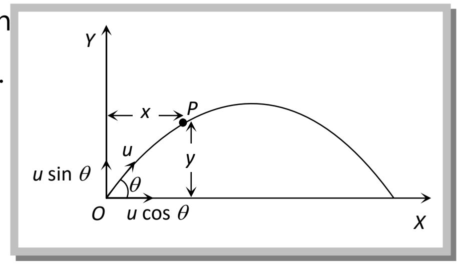
Oblique Projectile.

In projectile motion, horizontal component of velocity (u cosθ), acceleration (g) and mechanical energy remains constant while, speed, velocity, vertical component of velocity (u sin θ), momentum, kinetic energy and potential energy all changes. Velocity, and KE are maximum at the point of projection while minimum (but not zero) at highest point.

(1) **Equation of trajectory:** A projectile thrown with velocity u at an angle θ with the horizontal. The velocity u can be resolved into two rectangular components.

v cos θ component along X–axis and u sin θ component along Y–axis.



For horizontal motion x = u cosθ× t $\Rightarrow t = \dfrac{x}{u\cos\theta}$ .... (i)

For vertical motion $y = (u\sin\theta)t - \dfrac{1}{2}gt^2$ .... (ii)

From equation (i) and (ii) $y = u\sin\theta\left(\dfrac{x}{u\cos\theta}\right) - \dfrac{1}{2}g\left(\dfrac{x^2}{u^2\cos^2\theta}\right)$

$$y = x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2\cos^2\theta}$$

This equation shows that the trajectory of projectile is parabolic because it is similar to equation of parabola

$$y = ax - bx^2$$

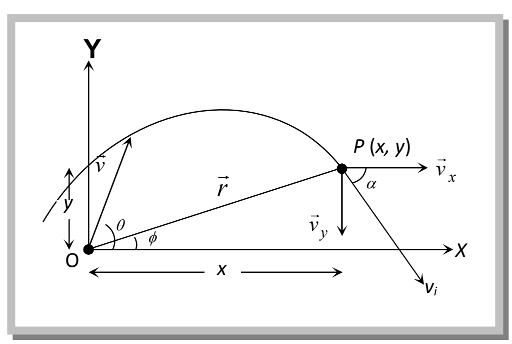
Note: Equation of oblique projectile also can be written as

$$y = x\tan\theta\left[1 - \frac{x}{R}\right]$$

(Where R = horizontal range = $\dfrac{u^2\sin 2\theta}{g}$)

(2) **Displacement of projectile $(\vec{r})$:** Let the particle acquires a position P having the coordinates (x, y) just after time t from the instant of projection. The corresponding position vector of the particle at time t is $\vec{r}$ as shown in the figure.

$$\vec{r} = x\hat{i} + y\hat{j} \quad ....(i)$$



The horizontal distance covered during time t is given as

$$x = v_x t \Rightarrow x = u\cos\theta\ t \quad ....(ii)$$

The vertical velocity of the particle at time t is given as

$$v_y = (v_0)_y - gt, \quad ....(iii)$$

Now the vertical displacement y is given as

$$y = u\sin\theta\, t - 1/2\, gt^2 \qquad \text{....(iv)}$$

Putting the values of x and y from equation (ii) and equation (iv) in equation (i) we obtain the position vector at any time t as

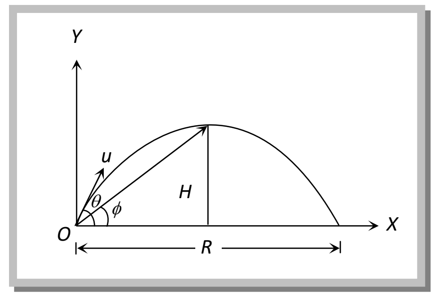
$$\vec{r} = (u\cos\theta)t\hat{i} + \left((u\sin\theta)t - \frac{1}{2}gt^2\right)\hat{j} \Rightarrow r = \sqrt{(u\,t\cos\theta)^2 + \left((u\,t\sin\theta) - \frac{1}{2}gt^2\right)^2}$$

$$r = u\,t\sqrt{1 + \left(\frac{gt}{2u}\right)^2 - \frac{gt\sin\theta}{u}}$$ and $\phi = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{ut\sin\theta - 1/2gt^2}{(u\,t\cos\theta)}\right)$ or

$$\phi = \tan^{-1}\left(\frac{2u\sin\theta - gt}{2u\cos\theta}\right)$$

Note: The angle of elevation ϕ of the highest point of the projectile and the angle of projection θ are related to each other as

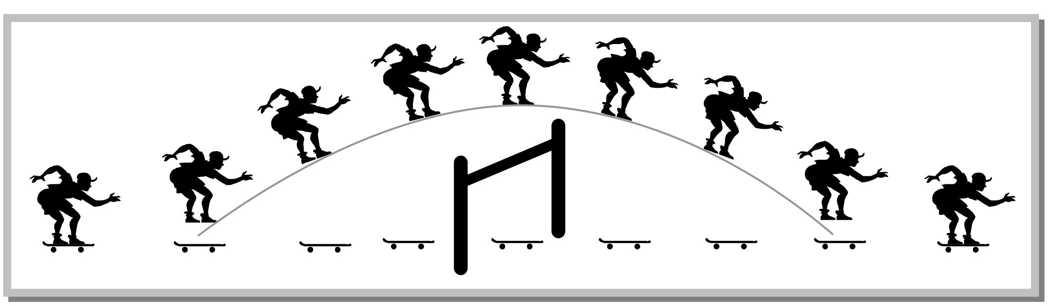
$$\tan\phi = \frac{1}{2}\tan\theta$$



(3) **Instantaneous velocity v:** In projectile motion, vertical component of velocity changes but horizontal component of velocity remains always constant.

Example: When a man jumps over the hurdle leaving behind its skateboard then vertical component of his velocity is changing, but not the horizontal component, which matches with the skateboard velocity.

As a result, the skateboard stays underneath him, allowing him to land on it.



Let v_i be the instantaneous velocity of projectile at time t direction of this velocity is along the tangent to the trajectory at point P.

$$\vec{v}_i = v_x i + v_y \hat{j} \Rightarrow v_i = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 \cos^2\theta + (u\sin\theta - gt)^2}$$

$$v_i = \sqrt{u^2 + g^2t^2 - 2u\,gt\sin\theta}$$

Direction of instantaneous velocity $\tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$ or

$$\alpha = \tan^{-1}\left[\tan\theta - \frac{gt}{u}\sec\theta\right]$$

(4) **Change in velocity:**Initial velocity (at projection point) $\vec{u}_i = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$

Final velocity (at highest point) $\vec{u}_f = u\cos\theta\,\hat{i} + 0\,\hat{j}$

(i) Change in velocity (Between projection point and highest point) $\Delta u = \vec{u}_f - \vec{u}_i = -u\sin\theta\,\hat{j}$

When body reaches the ground after completing its motion then final velocity $\vec{u}_f = u\cos\theta\,\hat{i} - u\sin\theta\,\hat{j}$

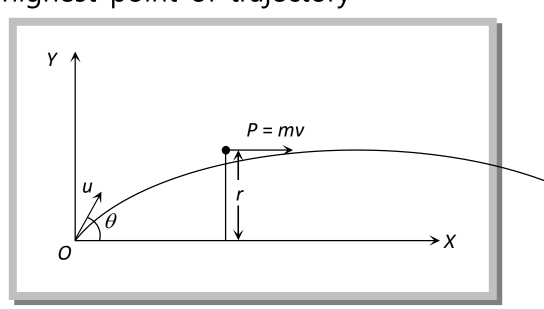
(ii) Change in velocity (Between complete projectile motions) $\Delta u = u_f - u_i = -2u\sin\theta\,\hat{i}$

(5) **Change in momentum:**Simply by the multiplication of mass in the above expression of velocity (Article-4).

(i) Change in momentum (Between projection point and highest point) $\Delta p = \vec{p}_f - \vec{p}_i = -mu\sin\theta\,\hat{j}$

(ii) Change in momentum (For the complete projectile motion) $\Delta p = \vec{p}_f - \vec{p}_i = -2mu\sin\theta\,\hat{j}$

(6) **Angular momentum:** Angular momentum of projectile at highest point of trajectory about the point of projection is given by



$$L = mvr \qquad \left[\text{Here } r = H = \frac{u^2\sin^2\theta}{2g}\right]$$

$$\therefore \qquad L = m\,u\cos\theta\frac{u^2\sin^2\theta}{2g} = \frac{m\,u^3\cos\theta\sin^2\theta}{2g}$$

(7) **Time of flight :** The total time taken by the projectile to go up and come down to the same level from which it was projected is called time of flight.

For vertical upward motion 0 = u sin θ – gt⇒ t = (u sin θ/g)

Now as time taken to go up is equal to the time taken to come down so

Time of flight $T = 2t = \frac{2u\sin\theta}{g}$

(i) Time of flight can also be expressed as: $T = \frac{2.u_y}{g}$ (where u_y is the vertical component of initial velocity).

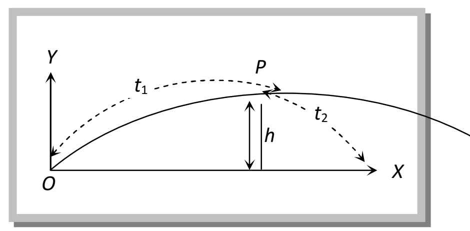
(ii) For complementary angles of projection θ and 90° – θ

(a) Ratio of time of flight = $\frac{T_1}{T_2} = \frac{2u\sin\theta/g}{2u\sin(90-\theta)/g} = \tan\theta \Rightarrow \frac{T_1}{T_2} = \tan\theta$

(b) Multiplication of time of flight = $T_1T_2 = \frac{2u\sin\theta}{g}\frac{2u\cos\theta}{g} \Rightarrow T_1T_2 = \frac{2R}{g}$

(iii) If t_1 is the time taken by projectile to rise up to point p and t_2 is the time taken in falling from point p to ground level then $t_1 + t_2 = \frac{2u\sin\theta}{g} =$ time of flight

or $u\sin\theta = \frac{g(t_1+t_2)}{2}$



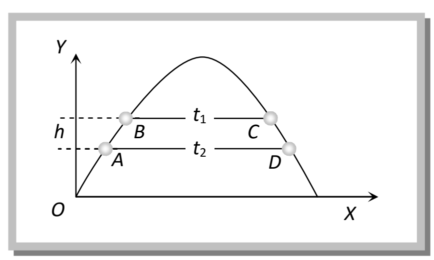
and height of the point p is given by $h = u\sin\theta\, t_1 - \frac{1}{2}gt_1^2$

$$h = g\frac{(t_1+t_2)}{2}t_1 - \frac{1}{2}gt_1^2$$

by solving $h = \frac{g\,t_1t_2}{2}$

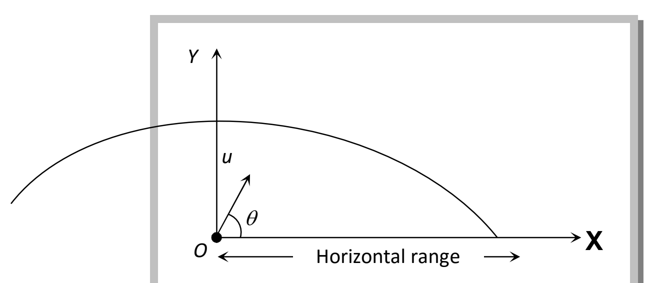
(iv) If B and C are at the same level on trajectory and the time difference between these two points is t_1, similarly A and D are also at the same level and the time difference between these two positions is t_2 then

$$t_2^2 - t_1^2 = \frac{8h}{g}$$



(8) **Horizontal range:**It is the horizontal distance travelled by a body during the time of flight.

So by using second equation of motion



$$R = u\cos\theta \times T = u\cos\theta \times (2u\sin\theta / g) = \frac{u^2 \sin 2\theta}{g}$$

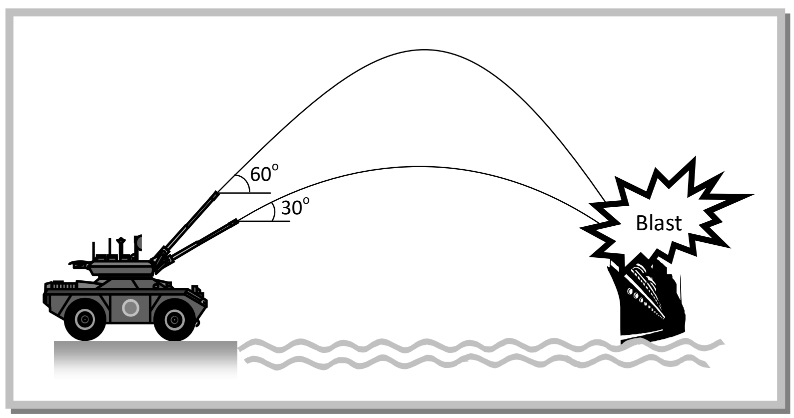
$$R = \frac{u^2 \sin 2\theta}{g}$$

(i) Range of projectile can also be expressed as:

$$R = u\cos\theta \times T = u\cos\theta \frac{2u\sin\theta}{g} = \frac{2u\cos\theta \; u\sin\theta}{g} = \frac{2u_x u_y}{g}$$

$\therefore R = \frac{2u_x u_y}{g}$ (Where u_x and u_y are the horizontal and vertical component of initial velocity)

(ii) If angle of projection is changed from θ to θ′ = (90 – θ) then range remains unchanged.



$$R' = \frac{u^2 \sin 2\theta'}{g} = \frac{u^2 \sin[2(90^o - \theta)]}{g} = \frac{u^2 \sin 2\theta}{g} = R$$

So a projectile has same range at angles of projection θ and (90 – θ), though time of flight, maximum height and trajectories are different.

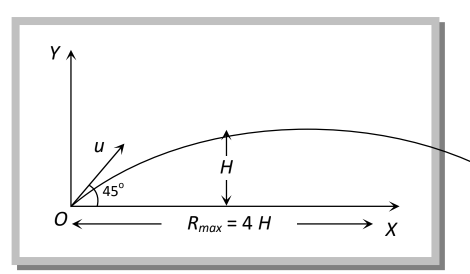
These angles θ and 90° – θ are called complementary angles of projection and for complementary angles of projection ratio of range $\frac{R_1}{R_2} = \frac{u^2 \sin 2\theta / g}{u^2 \sin[2(90^o - \theta)]/g} = 1 \Rightarrow \frac{R_1}{R_2} = 1$

(iii) For angle of projection θ_1 = (45 – α) and θ_2 = (45 + α), range will be same and equal to $u^2 \cos 2\alpha/g$.

θ_1 and θ_2 are also the complementary angles.

(iv) Maximumrange: For range to be maximum

$$\frac{dR}{d\theta} = 0 \Rightarrow \frac{d}{d\theta}\left[\frac{u^2 \sin 2\theta}{g}\right] = 0$$



$\Rightarrow \cos 2\theta = 0$ i.e. $2\theta = 90^o \Rightarrow \theta = 45^o$ and $R_{max} = (u^2/g)$

i.e., a projectile will have maximum range when it is projected at an angle of 45° to the horizontal and the maximum range will be (u^2/g).

When the range is maximum, the height H reached by the projectile

$$H = \frac{u^2 \sin^2 \theta}{2g} = \frac{u^2 \sin^2 45}{2g} = \frac{u^2}{4g} = \frac{R_{max}}{4}$$

i.e., if a person can throw a projectile to a maximum distance R_{max}, The maximum height to which it will rise is $\left(\frac{R_{max}}{4}\right)$.

(v) Relation between horizontal range and maximum height: $R = \frac{u^2 \sin 2\theta}{g}$ and $H = \frac{u^2 \sin^2 \theta}{2g}$

$$\therefore \frac{R}{H} = \frac{u^2 \sin 2\theta / g}{u^2 \sin^2 \theta / 2g} = 4 \cot \theta \qquad \Rightarrow R = 4H \cot \theta$$

(vi) If in case of projectile motion range R is n times the maximum height H

i.e. $R = nH \Rightarrow \frac{u^2 \sin 2\theta}{g} = n \frac{u^2 \sin^2 \theta}{2g} \Rightarrow \tan \theta = [4/n]$ or $\theta = \tan^{-1}[4/n]$

The angle of projection is given by $\theta = \tan^{-1}[4/n]$

Note : If R = H then $\theta = \tan^{-1}(4)$ or $\theta = 76^o$.

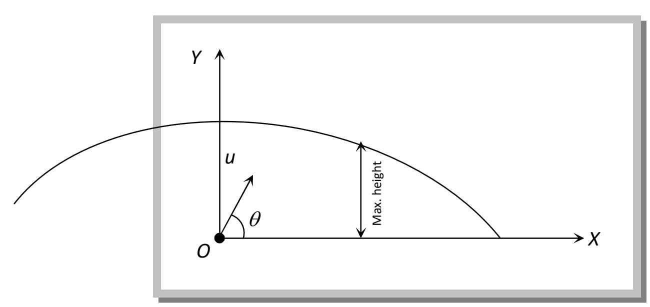
If R = 4H then $\theta = \tan^{-1}(1)$ or $\theta = 45^o$.

(9) **Maximum height:** It is the maximum height from the point of projection, a projectile can reach.

So, by using $v^2 = u^2 + 2as$

$$0 = (u \sin \theta)^2 - 2gH$$

$$H = \frac{u^2 \sin^2 \theta}{2g}$$



(i) Maximum height can also be expressed as

$H = \frac{u_y^2}{2g}$ (where u_y is the vertical component of initial velocity).

(ii) $H_{max} = \frac{u^2}{2g}$ (when $\sin^2\theta$ = max = 1 i.e., $\theta = 90^o$)

i.e., for maximum height body should be projected vertically upward. So it falls back to the point of projection after reaching the maximum height.

(iii) For complementary angles of projection θ and 90° – θ

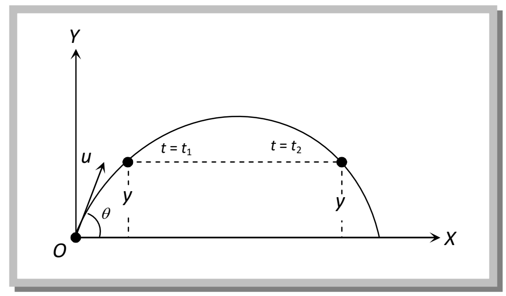
Ratio of maximum height = $\frac{H_1}{H_2} = \frac{u^2 \sin^2\theta/2g}{u^2 \sin^2(90^o - \theta)2g} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

$\therefore \frac{H_1}{H_2} = \tan^2\theta$

(10) **Projectile passing through two different points on same height at time t_1 and t_2:**

If the particle passes two points situated at equal height y at $t = t_1$ and $t = t_2$, then

(i) **Height (y):** $y = (u\sin\theta)t_1 - \frac{1}{2}gt_1^2$ .....(i)



and $y = (u\sin\theta)t_2 - \frac{1}{2}gt_2^2$ .....(ii)

Comparing equation (i) with equation (ii)

$$u\sin\theta = \frac{g(t_1+t_2)}{2}$$

Substituting this value in equation (i)

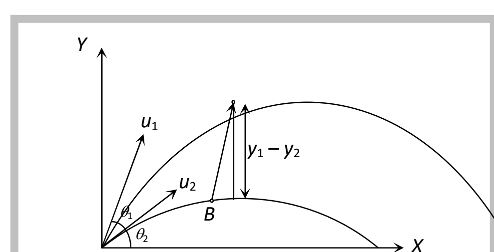
$$y = g\left(\frac{t_1+t_2}{2}\right)t_1 - \frac{1}{2}gt_1^2 \qquad \Rightarrow y = \frac{gt_1t_2}{2}$$

(ii) **Time (t_1 and t_2):** $y = u\sin\theta\, t - \frac{1}{2}gt^2$

$$t^2 - \frac{2u\sin\theta}{g}t + \frac{2y}{g} = 0 \qquad \Rightarrow t = \frac{u\sin\theta}{g}\left[1 \pm \sqrt{1 - \left(\frac{\sqrt{2gy}}{u\sin\theta}\right)^2}\right]$$

$$t_1 = \frac{u\sin\theta}{g}\left[1 + \sqrt{1 - \left(\frac{\sqrt{2gy}}{u\sin\theta}\right)^2}\right] \text{ and } t_2 = \frac{u\sin\theta}{g}\left[1 - \sqrt{1 - \left(\frac{\sqrt{2gy}}{u\sin\theta}\right)^2}\right]$$

(11) **Motion of a projectile as observed from another projectile:** Suppose two balls A and B are projected simultaneously from the origin, with initial velocities u_1 and u_2 at angle θ_1 and θ_2, respectively with the horizontal.



The instantaneous positions of the two balls are given by

Ball A : $x_1 = (u_1 \cos\theta_1)t$ $\quad y_1 = (u_1 \sin\theta_1)t - \frac{1}{2}gt^2$

Ball B : $x_2 = (u_2 \cos\theta_2)t$ $\quad y_2 = (u_2 \sin\theta_2)t - \frac{1}{2}gt^2$

The position of the ball A with respect to ball B is given by

$$x = x_1 - x_2 = (u_1 \cos\theta_1 - u_2 \cos\theta_2)t$$

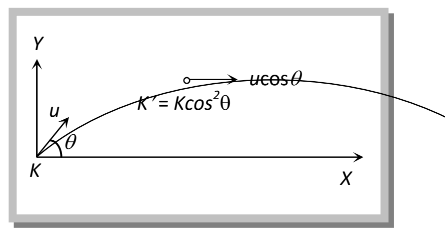
$$y = y_1 - y_2 = (u_1 \sin\theta_1 - u_2 \sin\theta_2)t$$

Now $\frac{y}{x} = \left(\frac{u_1 \sin\theta_1 - u_2 \sin\theta_2}{u_1 \cos\theta_1 - u_2 \cos\theta_2}\right) =$ constant

Thus motion of a projectile relative to another projectile is a straight line.

(12) **Energy of projectile:** When a projectile moves upward its kinetic energy decreases, potential energy increases but the total energy always remain constant.

If a body is projected with initial kinetic energy $K(=1/2\ mu^2)$, with angle of projection θ with the horizontal then at the highest point of trajectory



(i) **Kinetic energy** $= \frac{1}{2}m(u\cos\theta)^2 = \frac{1}{2}mu^2\cos^2\theta$

$\therefore\ K' = K\cos^2\theta$

(ii) **Potential energy** $= mgH = mg\frac{u^2\sin^2\theta}{2g} = \frac{1}{2}mu^2\sin^2\theta \left(\text{As } H = \frac{u^2\sin^2\theta}{2g}\right)$

(iii) **Total energy** = Kinetic energy + Potential energy $= \frac{1}{2}mu^2\cos^2\theta + \frac{1}{2}mu^2\sin^2\theta$

$= \frac{1}{2}mu^2$ = Energy at the point of projection.

This is in accordance with the law of conservation of energy.