Perfectly Elastic Head-On Collision:

Let two bodies of masses m_1 and m_2 moving with initial velocities u_1 and u_2 in the same direction and they collide such that after collision their final velocities are v_1 and v_2 respectively.

According to law of conservation of momentum

$$m_1u_1+m_2u_2=m_1v_1+m_2v_2 \quad \text{...(i)}$$

$$\Rightarrow m_1(u_1-v_1)=m_2(v_2-u_2) \quad \text{...(ii)}$$

According to law of conservation of kinetic energy

$$\frac{1}{2}{m}_{1}u_{1}^{2}+\frac{1}{2}{m}_{2}u_{2}^{2}=\frac{1}{2}{m}_{1}v_{1}^{2}+\frac{1}{2}{m}_{2}v_{2}^{2} \quad \text{...(iii)}$$

$$\Rightarrow m_1(u_1^2-v_1^2)=m_2(v_2^2-u_2^2) \quad \text{....(iv)}$$

Dividing equation (iv) by equation (ii)

$$v_1+u_1=v_2+u_2 \quad \text{...(v)}$$

$$\Rightarrow u_1-u_2=v_2-v_1 \quad \text{...(vi)}$$

Relative velocity of separation is equal to relative velocity of approach.

Note :

- The ratio of relative velocity of separation and relative velocity of approach is defined as coefficient of restitution.

$$e=\frac{v_2-v_1}{u_1-u_2}$$

or

$$v_2 - v_1 = e(u_1 - u_2)$$

For perfectly elastic collision, $e = 1$,

$\therefore \quad v_2 - v_1 = u_1 - u_2$

[As shown in eq. (vi)] For perfectly inelastic collision, $e = 0$

$\therefore \quad v_2 - v_1 = 0$

or $\quad v_2 = v_1$

It means that two body stick together and move with same velocity.

For inelastic collision,

$0 < e$
< 1

$\therefore \quad v_2 - v_1 = e(u_1 - u_2)$

In short we can say that e is the degree of elasticity of collision and it is dimensionless quantity.

Further from equation (v) we get

$v_2 = v_1 + u_1 - u_2$

Substituting this value of v_2 in equation (i) and rearranging, we get

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \frac{2m_2 u_2}{m_1 + m_2} \qquad \text{...(vii)}$$

Similarly we get,

$$v_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2 + \frac{2m_1 u_1}{m_1 + m_2} \qquad \text{...(viii)}$$

(1) Special cases of head on elastic collision:

(i) If projectile and target are of same mass i.e.

$$m_1 = m_2$$

Since

$$\upsilon_1=\left(\frac{m_1-m_2}{m_1+m_2}\right)u_1+\frac{2m_2}{m_1+m_2}u_2$$

and

$$\upsilon_2=\left(\frac{m_2-m_1}{m_1+m_2}\right)u_2+\frac{2m_1u_1}{m_1+m_2}$$

Substituting $m_1=m_2$, we get

$$\upsilon_1=u_2$$

and

$$\upsilon_2=u_1$$

It means when two bodies of equal masses undergo head on elastic collision, their velocities get interchanged.

Example : Collision of two billiard balls

(ii) If massive projectile collides with a light target i.e.

$$m_1>>m_2$$

Since

$$\upsilon_1=\left(\frac{m_1-m_2}{m_1+m_2}\right)u_1+\frac{2m_2u_2}{m_1+m_2}$$

and

$$\upsilon_2=\left(\frac{m_2-m_1}{m_1+m_2}\right)u_2+\frac{2m_1u_1}{m_1+m_2}$$

Substituting $m_2=0$, we get

$$\upsilon_1=u_1$$

and

$$\upsilon_2=2u_1-u_2$$

Example: Collision of a truck with a cyclist.

(iii) If light projectile collides with a very heavy target i.e.

$$m_1<<m_2$$

Since

$$\upsilon_1=(m_1-m_2m_1+m_2)u_1+2m_2u_2m_1+m_2$$

and

$$\upsilon_2=(m_2-m_1m_1+m_2)u_2+2m_1u_1m_1+m_2$$

Substituting

$m_1=0$, we get

$$\upsilon_1=-u_1+2u_2 \quad \text{and} \quad \upsilon_2=u_2$$

Example: Collision of a ball with a massive wall.

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(2) Kinetic energy transfer during head on elastic collision

Kinetic energy of projectile before collision

$$K_i=12m_1u_{21}$$

Kinetic energy of projectile after collision

$$K_f=12m_1v_{21}$$

Kinetic energy transferred from projectile to target

$\Delta K=$ decrease in kinetic energy in projectile

$$\Delta K=12m_1u_{21}-12m_1v_{21}$$

$$=12m_1(u_{21}-v_{21})$$

Fractional decrease in kinetic energy

$$\Delta KK=_{12}m_1(u_{21}-v_{21})_{12}m_1u_{21}$$

$$=1-(v_1u_1)_2 \qquad \text{...(i)}$$

We can substitute the value of v_1 from the equation

$$v_1=(m_1-m_2m_1+m_2)u_1+2m_2u_2m_1+m_2$$

If the target is at rest i.e.

$$u_2=0$$

then

$$v_1=(m_1-m_2m_1+m_2)u_1$$

From equation (i)

$$\Delta KK=1-(m_1-m_2m_1+m_2)_2 \qquad \text{...(ii)}$$

or

$$\Delta KK=4m_1m_2(m_1+m_2)_2 \qquad \text{...(iii)}$$

or

$$\Delta KK=4m_1m_2(m_1-m_2)_2+4m_1m_2 \qquad \text{...(iv)}$$

Note :

- Greater the difference in masses, lesser will be transfer of kinetic energy and vice versa
- Transfer of kinetic energy will be maximum when the difference in masses is minimum

i.e.

$$m_1-m_2=0$$

or

$$m_1=m_2$$

then

$$\Delta KK=1=100$$

So the transfer of kinetic energy in head on elastic collision (when target is at rest) is maximum when the masses of particles are equal i.e. mass ratio is 1 and the transfer of kinetic energy is 100%.

- If

$$m_2=nm_1$$

then from equation (iii) we get

$$\Delta KK=4n(1+n)_2$$

- Kinetic energy retained by the projectile

$$(\Delta KK)_{Retained}=1-\text{ kinetic energy transferred by projectile}$$

$$\Rightarrow \quad (\Delta KK)_{Retained}=1-[1-(m_1-m_2m_1+m_2)_2] = (m_1-m_2m_1+m_2)_2$$

(3) Velocity, momentum and kinetic energy of stationary target after head on elastic collision

(i) Velocity of target: We know

$$v_2=(m_2-m_1m_1+m_2)u_2+2m_1u_1m_1+m_2$$

$\Rightarrow \quad v_2=2m_1u_1m_1+m_2$

$$=2u_11+m_2/m_1$$

As $\upsilon_2=0$ and Assuming $m_2m_1=n$

$\therefore \; v_2=2u_11+n$

(ii) Momentum of target :

$$P_2=m_2v_2= 2nm_1u_11+n$$

$$[\text{As } m_2=m_1n \text{ and } v_2=2u_11+n]$$

$\therefore \; P_2=2m_1u_11+(1/n)$

(iii) Kinetic energy of target :

$$K_2=12m_2v_{22}$$

$$=12nm_1(2u_11+n)_2$$

$$=2m_1u_{21}n(1+n)_2$$

$$=4(K_1)n(1-n)_2+4n$$

$$[\text{ As } K_1=12m_1u_{21}]$$

(iv) Relation between masses for maximum velocity, momentum and kinetic energy

Velocity	$\upsilon_2=2u_11+n$	For υ_2 to be maximum n must be minimum i.e. $n=m_2m_1\rightarrow 0$ $\therefore$ $m_2<<m_1$	Target should be very light.
Momentum	$P_2=2m_1u_1(1+1/n)$	For P_2 to be maximum, (1/n) must be minimum or n must be maximum. i.e. $n=m_2m_1\rightarrow\infty$ $\therefore$ $m_2>>m_1$	Target should be massive.
Kinetic energy	$K_2=4K_1n(1-n)_2+4n$	For K_2	Target and projectile should be of equal mass.

		to be maximum $(1-n)_2$ must be minimum. i.e. $1-n=0 \Rightarrow n=1=m_2 m_1$ $\therefore$ $m_2=m_1$	